New (n, r)-arcs in PG(2, 17) and PG(2, 19) *

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Abstract - An (n, r)-arc is a set of n points of a projective plane such that some r, but no $r+1$ of them, are collinear. In this paper new $(95, 7)$-arc, $(183, 12)$-arc, $(205, 13)$-arc in PG(2,17) and $(243, 14)$-arc, $(264, 15)$-arc in PG(2,19) are constructed.

1 Introduction

Let GF(q) denote the Galois field of q elements and V(3, q) be the vector space of row vectors of length three with entries in GF(q). Let PG(2, q) be the corresponding projective plane. The *points* (x_1, x_2, x_3) of PG(2,q) are the 1-dimensional subspaces of V(3,q). Subspaces of dimension two are called *lines*. The number of points and the number of lines in PG(2, q) is q^2+q+1. There are $q+1$ points on every line and $q+1$ lines through every point.

Definition 1.1 *An (n, r)-arc is a set of n points of a projective plane such that some r, but no $r+1$ of them, are collinear.*

Definition 1.2 *The maximum size of a (k, r)-arc in PG(2, q) is denoted by $m_r(2, q)$.*

Definition 1.3 *Let M be a set of points in any plane. An i-secant is a line meeting M in exactly i points. Define τ_i as the number of i-secants to a set M.*

In terms of τ_i the definition of (n, r)-arc becomes

Definition 1.4 *An (n, r)-arc is a set of n points of a projective plane for which $\tau_i \geq 0$ for $i < r$, $\tau_r > 0$ and $\tau_i = 0$ when $i > r$.*

Definition 1.5 *An (l, t)-blocking set S in* PG(2, q) *is a set of l points such that every line of* PG(2, q) *intersects S in at least t points, and there is a line intersecting S in exactly t points.*

*Supported in part by the Bulgarian Ministry of Education and Science under contract in TU-Gabrovo.

Note that an (n, r)-arc is the complement of a $(q^2 + q + 1 - n, q + 1 - r)$-blocking set in a projective plane and conversely.

The following two theorems are proved in [1] and [2] respectively.

Theorem 1.1 *Let K be an (n, r)-arc in* PG(2, q) *where q is prime.*

1. *If $r \le (q + 1)/2$ then $m_r(2, q) \le (r - 1)q + 1$.*
2. *If $r \ge (q + 3)/2$ then $m_r(2, q) \le (r - 1)q + r - (q + 1)/2$.*

Theorem 1.2 *Let K be a (k, r)-arc in* PG(2, q) *with $r > (q + 3)/2$ and $q \le 29$ is prime. Then*

$$m_r(2, q) \le (r - 1)q + r - (q + 3)/2.$$

The right values of $m_r(2, q)$ are known only for $q \le 9$ (see [3], [4]).

Values of $m_r(2, q)$

r\q	3	4	5	7	8	9
2	4	6	6	8	10	10
3		9	11	15	15	17
4			16	22	28	28
5				29	33	37
6				36	42	48
7					49	55
8						65

It is proved in [5] that $m_{12}(2, 17) \ge 182$. In [6] it is established that $m_7(2, 17) \ge 94, m_{13}(2, 17) \ge 204$, $m_{14}(2, 19) \ge 242, m_{15}(2, 19) \ge 262$. These bounds are the best known until now (see [4]). In this paper we will improve the first four bounds with one and the last bound with two.

The elements of GF(17) are denoted by $0, 1, 2, \ldots, 9$, $10 = a$, $11 = b$, $12 = c$, $13 = d$, $14 = e$, $15 = f$, $16 = g$. The elements of GF(19) are denoted by $0, 1, 2, \ldots, 9$, $10 = a$, $11 = b$, $12 = c$, $13 = d$, $14 = e$, $15 = f$, $16 = g$, $17 = h$, $18 = i$.

2 New arcs in PG(2, 17)

Theorem 2.1 *There exist a $(95, 7)$-arc, a $(183, 12)$-arc and a $(205, 13)$-arc in* PG(2,17). *Therefore,*

$$95 \le m_7(2, 17) \le 103, \quad 183 \le m_{12}(2, 17) \le 189, \quad 205 \le m_{13}(2, 17) \le 207.$$

Proof:
1. The set of points

$$\begin{array}{llllllll}
(1,1,3), & (1,1,6), & (1,1,8), & (1,1,b), & (1,1,e), & (1,1,f), & (1,2,2), & (1,2,3),\\
(1,2,5), & (1,2,c), & (1,2,f), & (1,3,0), & (1,3,3), & (1,3,5), & (1,3,6), & (1,3,b),\\
(1,3,e), & (1,3,f), & (1,4,1), & (1,4,4), & (1,4,6), & (1,4,b), & (1,4,c), & (1,4,d),\\
(1,4,g), & (1,5,0), & (1,5,1), & (1,5,3), & (1,5,6), & (1,5,7), & (1,5,e), & (1,5,g),\\
(1,6,1), & (1,6,2), & (1,6,4), & (1,6,5), & (1,6,d), & (1,6,f), & (1,6,g), & (1,7,2),\\
(1,7,4), & (1,7,7), & (1,7,8), & (1,7,9), & (1,7,a), & (1,7,d), & (1,8,6), & (1,8,7),\\
(1,8,b), & (1,8,d), & (1,9,5), & (1,9,6), & (1,9,7), & (1,9,9), & (1,9,a), & (1,9,b),\\
(1,9,g), & (1,a,0), & (1,a,2), & (1,a,6), & (1,a,8), & (1,a,b), & (1,a,f), & (1,a,g),\\
(1,b,0), & (1,b,3), & (1,b,4), & (1,b,e), & (1,b,g), & (1,c,0), & (1,c,4), & (1,c,5),\\
(1,c,7), & (1,c,b), & (1,c,c), & (1,d,3), & (1,d,4), & (1,d,7), & (1,d,a), & (1,d,d),\\
(1,d,e), & (1,d,g), & (1,e,0), & (1,e,3), & (1,e,5), & (1,e,7), & (1,e,9), & (1,e,a),\\
(1,e,c), & (1,f,5), & (1,f,8), & (1,f,9), & (1,f,c), & (1,f,d), & (1,f,e), &
\end{array}$$

forms a $(95,7)$-arc in PG(2, 17) with secant distribution

$$\tau_0 = 21, \quad \tau_1 = 1, \quad \tau_2 = 3, \quad \tau_3 = 9, \quad \tau_4 = 23, \quad \tau_5 = 43, \quad \tau_6 = 80, \quad \tau_7 = 127.$$

2. The set of points

$$\begin{array}{llllllll}
(0,0,1), & (0,1,1), & (0,1,2), & (0,1,7), & (0,1,a), & (0,1,g), & (1,0,1), & (1,0,5),\\
(1,0,7), & (1,0,a), & (1,0,g), & (1,1,6), & (1,1,7), & (1,1,8), & (1,1,9), & (1,1,b),\\
(1,2,5), & (1,2,8), & (1,2,9), & (1,2,c), & (1,2,d), & (1,2,f), & (1,3,3), & (1,3,4),\\
(1,3,5), & (1,3,7), & (1,3,a), & (1,3,b), & (1,3,d), & (1,3,e), & (1,4,0), & (1,4,1),\\
(1,4,3), & (1,4,6), & (1,4,7), & (1,4,8), & (1,4,a), & (1,4,b), & (1,4,c), & (1,4,d),\\
(1,4,e), & (1,5,0), & (1,5,2), & (1,5,5), & (1,5,c), & (1,5,e), & (1,5,f), & (1,6,0),\\
(1,6,1), & (1,6,4), & (1,6,7), & (1,6,9), & (1,6,a), & (1,6,b), & (1,6,c), & (1,6,d),\\
(1,6,g), & (1,7,0), & (1,7,2), & (1,7,3), & (1,7,e), & (1,7,g), & (1,8,1), & (1,8,2),\\
(1,8,4), & (1,8,6), & (1,8,d), & (1,8,f), & (1,8,g), & (1,9,1), & (1,9,2), & (1,9,6),\\
(1,9,a), & (1,9,b), & (1,9,d), & (1,9,f), & (1,9,g), & (1,a,0), & (1,a,1), & (1,a,3),\\
(1,a,e), & (1,a,g), & (1,b,1), & (1,b,4), & (1,b,5), & (1,b,6), & (1,b,8), & (1,b,9),\\
(1,b,a), & (1,b,d), & (1,b,g), & (1,c,0), & (1,c,2), & (1,c,5), & (1,c,c), & (1,c,f),\\
(1,d,3), & (1,d,5), & (1,d,6), & (1,d,7), & (1,d,b), & (1,d,c), & (1,d,e), & (1,e,4),\\
(1,e,5), & (1,e,7), & (1,e,a), & (1,e,c), & (1,e,d), & (1,e,e), & (1,e,g), & (1,f,1),\\
(1,f,2), & (1,f,4), & (1,f,5), & (1,f,8), & (1,f,9), & (1,f,c), & (1,f,f), & (1,g,3),\\
(1,g,6), & (1,g,8), & (1,g,9), & (1,g,b), & & & &
\end{array}$$

forms a $(124,6)$-blocking set in PG(2, 17) with secant distribution

$$\tau_6 = 124, \quad \tau_7 = 93, \quad \tau_8 = 53, \quad \tau_9 = 23, \quad \tau_{10} = 3, \quad \tau_{11} = 2, \quad \tau_{12} = 1, \quad \tau_{17} = 2, \quad \tau_{18} = 6.$$

The complement of this blocking set is a $(183,12)$-arc in PG(2, 17).

3. The set of points

$$\begin{array}{llllllll}
(0,0,1), & (0,1,0), & (0,1,1), & (0,1,4), & (0,1,d), & (0,1,g), & (1,0,0), & (1,0,1), \\
(1,0,4), & (1,0,d), & (1,0,g), & (1,1,0), & (1,1,1), & (1,1,4), & (1,1,d), & (1,2,4), \\
(1,2,8), & (1,2,9), & (1,2,d), & (1,3,4), & (1,3,5), & (1,3,c), & (1,3,d), & (1,4,0), \\
(1,4,1), & (1,4,2), & (1,4,3), & (1,4,4), & (1,4,5), & (1,4,6), & (1,4,7), & (1,4,8), \\
(1,4,9), & (1,4,a), & (1,4,b), & (1,4,c), & (1,4,d), & (1,4,e), & (1,4,f), & (1,4,g), \\
(1,5,3), & (1,5,4), & (1,5,d), & (1,5,e), & (1,6,4), & (1,6,7), & (1,6,a), & (1,6,d), \\
(1,7,4), & (1,7,6), & (1,7,b), & (1,7,d), & (1,8,2), & (1,8,4), & (1,8,d), & (1,8,f), \\
(1,9,2), & (1,9,4), & (1,9,d), & (1,9,f), & (1,a,4), & (1,a,6), & (1,a,b), & (1,a,d), \\
(1,b,4), & (1,b,7), & (1,b,a), & (1,b,d), & (1,c,3), & (1,c,4), & (1,c,d), & (1,c,e), \\
(1,d,0), & (1,d,1), & (1,d,2), & (1,d,3), & (1,d,4), & (1,d,5), & (1,d,6), & (1,d,7), \\
(1,d,8), & (1,d,9), & (1,d,a), & (1,d,b), & (1,d,c), & (1,d,d), & (1,d,e), & (1,d,f), \\
(1,d,g), & (1,e,4), & (1,e,5), & (1,e,c), & (1,e,d), & (1,f,4), & (1,f,8), & (1,f,9), \\
(1,f,d), & (1,g,0), & (1,g,1), & (1,g,4), & (1,g,d), & (1,g,g), & &
\end{array}$$

forms a $(102,5)$-blocking set in PG(2, 17) with secant distribution

$$\tau_5 = 165, \quad \tau_6 = 66, \quad \tau_7 = 57, \quad \tau_8 = 9, \quad \tau_9 = 4, \quad \tau_{18} = 6.$$

The complement of this blocking set is a $(205,13)$-arc in PG(2, 17).

3 New arcs in PG(2, 19)

Theorem 3.1 *There exist a* $(243,14)$*-arc and a* $(264,15)$*-arc in* PG(2,19)*. Therefore,*

$$243 \le m_{14}(2,19) \le 250, \quad 264 \le m_{15}(2,19) \le 270.$$

Proof:
1. The set of points

$$\begin{array}{llllllll}
(0,1,0), & (0,1,3), & (0,1,7), & (0,1,8), & (0,1,b), & (0,1,c), & (0,1,g), & (1,0,0), \\
(1,0,3), & (1,0,7), & (1,0,8), & (1,0,b), & (1,0,c), & (1,0,g), & (1,1,0), & (1,1,1), \\
(1,1,4), & (1,1,6), & (1,1,d), & (1,1,f), & (1,1,i), & (1,2,3), & (1,2,4), & (1,2,9), \\
(1,2,a), & (1,2,f), & (1,2,g), & (1,3,2), & (1,3,6), & (1,3,7), & (1,3,c), & (1,3,d), \\
(1,3,h), & (1,4,1), & (1,4,4), & (1,4,6), & (1,4,9), & (1,4,a), & (1,4,d), & (1,4,f), \\
(1,4,i), & (1,5,1), & (1,5,5), & (1,5,7), & (1,5,9), & (1,5,a), & (1,5,c), & (1,5,e), \\
(1,5,i), & (1,6,2), & (1,6,3), & (1,6,4), & (1,6,f), & (1,6,g), & (1,6,h), & (1,7,1), \\
(1,7,5), & (1,7,6), & (1,7,8), & (1,7,b), & (1,7,d), & (1,7,e), & (1,7,i), & (1,8,0),
\end{array}$$

$(1,8,2)$, $(1,8,5)$, $(1,8,8)$, $(1,8,b)$, $(1,8,e)$, $(1,8,h)$, $(1,9,3)$, $(1,9,5)$,
$(1,9,8)$, $(1,9,b)$, $(1,9,e)$, $(1,9,g)$, $(1,a,3)$, $(1,a,5)$, $(1,a,8)$, $(1,a,b)$,
$(1,a,e)$, $(1,a,g)$, $(1,b,0)$, $(1,b,2)$, $(1,b,5)$, $(1,b,8)$, $(1,b,b)$, $(1,b,e)$,
$(1,b,h)$, $(1,c,1)$, $(1,c,5)$, $(1,c,6)$, $(1,c,8)$, $(1,c,b)$, $(1,c,d)$, $(1,c,e)$,
$(1,c,i)$, $(1,d,2)$, $(1,d,3)$, $(1,d,4)$, $(1,d,f)$, $(1,d,g)$, $(1,d,h)$, $(1,e,1)$,
$(1,e,5)$, $(1,e,7)$, $(1,e,9)$, $(1,e,a)$, $(1,e,c)$, $(1,e,e)$, $(1,e,i)$, $(1,f,1)$,
$(1,f,4)$, $(1,f,6)$, $(1,f,9)$, $(1,f,a)$, $(1,f,d)$, $(1,f,f)$, $(1,f,i)$, $(1,g,2)$,
$(1,g,6)$, $(1,g,7)$, $(1,g,c)$, $(1,g,d)$, $(1,g,h)$, $(1,h,3)$, $(1,h,4)$, $(1,h,9)$,
$(1,h,a)$, $(1,h,f)$, $(1,h,g)$, $(1,i,0)$, $(1,i,1)$, $(1,i,4)$, $(1,i,6)$, $(1,i,d)$,
$(1,i,f)$, $(1,i,i)$,

forms a $(138,6)$-blocking set in PG(2, 193) with secant distribution

$$\tau_6 = 137, \quad \tau_7 = 144, \quad \tau_8 = 64, \quad \tau_9 = 22, \tau_{10} = 6, \quad \tau_{20} = 8.$$

The complement of this blocking set is a $(243,14)$-arc in PG(2, 19).

2. The set of points

$(0,1,0)$, $(0,1,1)$, $(0,1,3)$, $(0,1,g)$, $(0,1,i)$, $(1,0,3)$, $(1,0,7)$, $(1,0,d)$,
$(1,0,g)$, $(1,0,i)$, $(1,1,0)$, $(1,1,6)$, $(1,1,c)$, $(1,1,d)$, $(1,1,e)$, $(1,2,0)$,
$(1,2,3)$, $(1,2,7)$, $(1,2,9)$, $(1,2,a)$, $(1,2,b)$, $(1,2,f)$, $(1,2,g)$, $(1,3,6)$,
$(1,3,7)$, $(1,3,a)$, $(1,3,c)$, $(1,3,d)$, $(1,3,g)$, $(1,4,4)$, $(1,4,9)$, $(1,4,a)$,
$(1,4,f)$, $(1,4,h)$, $(1,5,1)$, $(1,5,7)$, $(1,5,8)$, $(1,5,c)$, $(1,5,i)$, $(1,6,0)$,
$(1,6,2)$, $(1,6,4)$, $(1,6,7)$, $(1,6,f)$, $(1,6,h)$, $(1,7,1)$, $(1,7,5)$, $(1,7,6)$,
$(1,7,e)$, $(1,7,i)$, $(1,8,2)$, $(1,8,5)$, $(1,8,7)$, $(1,8,8)$, $(1,8,b)$, $(1,8,h)$,
$(1,9,3)$, $(1,9,4)$, $(1,9,5)$, $(1,9,8)$, $(1,9,b)$, $(1,9,e)$, $(1,9,g)$, $(1,a,3)$,
$(1,a,4)$, $(1,a,5)$, $(1,a,8)$, $(1,a,b)$, $(1,a,e)$, $(1,a,g)$, $(1,b,2)$, $(1,b,5)$,
$(1,b,7)$, $(1,b,8)$, $(1,b,9)$, $(1,b,b)$, $(1,b,h)$, $(1,c,1)$, $(1,c,5)$, $(1,c,6)$,
$(1,c,e)$, $(1,c,i)$, $(1,d,0)$, $(1,d,2)$, $(1,d,4)$, $(1,d,7)$, $(1,d,d)$, $(1,d,f)$,
$(1,d,h)$, $(1,e,1)$, $(1,e,7)$, $(1,e,8)$, $(1,e,c)$, $(1,e,i)$, $(1,f,4)$, $(1,f,9)$,
$(1,f,a)$, $(1,f,f)$, $(1,f,h)$, $(1,g,6)$, $(1,g,7)$, $(1,g,a)$, $(1,g,c)$, $(1,g,d)$,
$(1,g,g)$, $(1,h,3)$, $(1,h,7)$, $(1,h,9)$, $(1,h,a)$, $(1,h,b)$, $(1,h,f)$, $(1,h,g)$,
$(1,i,0)$, $(1,i,6)$, $(1,i,c)$, $(1,i,d)$, $(1,i,e)$,

forms a $(117,5)$-blocking set in PG(2, 19) with secant distribution

$$\tau_5 = 155, \quad \tau_6 = 124, \quad \tau_7 = 77, \quad \tau_8 = 13, \quad \tau_9 = 4, \quad \tau_{10} = 1, \quad \tau_{12} = 1, \quad \tau_{20} = 6.$$

The complement of this blocking set is a $(264,15)$-arc in PG(2, 19).

We can now update the table from [4] with the new bounds.

Bounds on $m_r(2, q)$

r\q	11	13	16	17	19
2	12	14	18	18	20
3	21	23	28..33	28..35	31..39
4	32..34	38..40	52	48..52	52..58
5	43..45	49..53	65	61..69	68..77
6	56	64..66	78..82	78..86	86..96
7	67	79	93..97	**95**..103	105..115
8	77..78	92	120	114..120	124..134
9	89..90	105	128..131	137	147..153
10	100..102	118..119	142..148	154	172
11		132..133	159..164	166..171	191
12		145..147	180..181	**183**..189	204..210
13			195..199	**205**..207	225..230
14			210..214	221..225	**243**..250
15			231	239..243	**264**..270
16				256..261	285..290
17					305..310
18					324..330
19					
20					
21					
22					

A close relationship between (n, r)-arcs in PG(2, q) and $[n, 3, d]_q$ codes is given in the next theorem.

Theorem 3.2 [7] *There exist a projective $[n, 3, d]_q$ code if and only if there exist an $(n, n-d)$-arc in* PG(2, q).

So there exist projective codes with parameters: $[95, 3, 88]_{17}$, $[183, 3, 171]_{17}$, $[205, 3, 192]_{17}$, $[243, 3, 229]_{19}$, $[264, 3, 249]_{19}$. All of these codes are Griesmer's codes.

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