

ARCS IN FINITE PROJECTIVE SPACES

SIMEON BALL, MICHEL LAVRAUW

ABSTRACT. This is an expository article detailing results concerning large arcs in finite projective spaces, which attempts to cover the most relevant results on arcs, simplifying and unifying proofs of known old and more recent theorems. The article is mostly self-contained and includes a proof of the most general form of Segre's lemma of tangents and a short proof of the MDS conjecture over prime fields based on this lemma.

CONTENTS

1. Introduction	2
2. Basic objects and definitions	3
3. Normal rational curves	3
4. Examples of large arcs in $\text{PG}(k-1, q)$	5
5. The trivial upper bound and the MDS conjecture	7
6. The projection theorem	9
7. The lemma of tangents	10
8. A system of equations associated with an arc	13
9. A proof of the MDS conjecture for prime fields	14
10. A proof of the MDS conjecture for $k \leq 2p-2$	15
11. The classification of the largest arcs for $k \leq p$ and $k \neq \frac{1}{2}(q+1)$	16
12. Extending small arcs to large arcs	17
13. The Segre-Blokhuis-Bruen-Thas hypersurface	18
14. A tensor associated to an arc	20
15. Planar arcs	21
16. A proof of the MDS conjecture for $k \leq \sqrt{q} - \sqrt{q}/p + 2$	25
17. Arcs in $\text{PG}(3, q)$	25
18. Arcs in $\text{PG}(4, q)$	26
19. State of the art on the MDS conjecture	27
References	29

Date: 23 November 2019.

The first author acknowledges the support of the project MTM2017-82166-P of the Spanish *Ministerio de Economía y Competitividad*. The second author acknowledges the support of *The Scientific and Technological Research Council of Turkey*, TÜBİTAK (project no. 118F159).

1. INTRODUCTION

An *arc* in a $(k-1)$ -dimensional projective space is a set of points with the property that any k of them span the whole space. An arc in a projective plane is called a *planar arc*. Arcs in finite projective spaces have a long history dating back at least as far as the 1950's and 1960's when Beniamino Segre published his fundamental work on arcs, including the articles [52], [53], [54], [56] and [57]. Segre's work included a proof of a conjecture of Järnefelt and Kustaanheimo [37] that a planar arc of size $q+1$, q odd, is a conic. See [53, footnote (5) p. 359] for an interesting historical account of this result.

Segre highlighted three problems concerning arcs. One of these problems (*Problema II_{r,q}* in [53]) is whether there exist parameters (k, q) for which every arc of size $q+1$ is a normal rational curve. This was answered in the affirmative by Segre's theorem for $k=3$ and q odd, but a full classification of such parameters has not been achieved yet. A second problem (*Problema III_{r,q}* in [53]) concerned the extendibility of arcs in projective spaces with such parameters, and asks for the possible size of an arc which is not extendable to a normal rational curve. Both of these problems have been the subject of much research in the last sixty years, and some of the main results are collected in this article. However, the most fundamental problem (*Problema I_{r,q}* in [53]) is that of determining the size and the structure of the largest arcs in a given projective space over a finite field. We still do not have a complete answer to this question, although progress is continuing to be made, and this article aims to provide unified proofs of the main results (including Segre's theorem) and the mathematics which was developed in the process. A solution to this problem would settle the famous *MDS conjecture* for maximum distance separable codes, see Section 5.

A simple combinatorial approach to these problems related to arcs reveals little. It is possible to improve on the trivial upper bound of $q+k-1$ for the size slightly but any reasonable results on arcs in k -dimensional projective spaces over a finite field with q elements, require an algebraic and geometric approach. Indeed, all the strongest results on arcs are based on Segre's initial ideas to find connections between arcs and algebraic curves/varieties.

Segre's approach relies on his lemma of tangents which will form the basis of this exposition. Note that in this context a *tangent* to a planar arc \mathcal{A} is a combinatorial tangent, i.e. a line meeting the arc \mathcal{A} in exactly one point, and this should not be confused with the notion of a tangent line of an algebraic curve. For a planar arc \mathcal{A} of size $q+1$, q odd, the lemma of tangents says that the triangle, formed by any three distinct points of \mathcal{A} , and the triangle, formed by the tangent lines at these points, are in perspective ([52, Theorem II]).

The most significant methodological contribution in Segre's work was to associate an algebraic curve to a planar arc. In particular, he proved that the set of tangents to a planar arc belong to an algebraic envelope of class t when q is even and of class $2t$ when q is odd ([55, Theorem I, II]), where the constant t is the number of tangents through a point of the arc. The crucial part in the proof of the existence of this envelope is [55, Theorem I], which can be seen as a generalisation of the theorem of Menelaos (which gives a simple condition for collinearity of three points on the sides of a triangle). Segre's envelope associated to a planar arc allowed the use of existing bounds on the number of points on algebraic curves over finite fields to obtain bounds for the size of planar arcs by careful analysis of its irreducible components, see for example [62], [32], [33] and [22].

More recently, elaborating on Segre's original ideas, reformulations and generalisations of his lemma of tangents have simplified the lemma which has extended its applicability. We refer to Section 7 for the *Coordinate-free lemma of tangents* from [1] and the most recent *Scaled coordinate-free lemma of tangents* from [6]. These lemmas have led to significant advancement in our knowledge of large arcs, not least that we now know how large an arc can be when q is prime [1]. A short proof of this result can be found in Section 9. The main motivation of the article was to set down results that can be derived using the afore-mentioned versions of Segre's lemma of tangents, which allow us to simplify proofs of many recent and old results.

Arcs appear in many branches of mathematics. In problems of finite geometry itself they are prevalent. To represent a matroid of rank r over a finite field one must map subsets of the matroid whose r -subsets are independent sets onto an arc, so the relevance of arcs to the representability of matroids is evident, see [26]. Non-existence results on arcs translate to non-representability results for matroids. An application of arcs to the theory error correcting codes, is given by the well-established fact that a k -dimensional linear maximum distance separable code is entirely equivalent to an arc in $\text{PG}(k-1, q)$. This is detailed in Section 5. The famous MDS conjecture, which is a focus of this article, comes from this view of arcs. MDS codes are the codes of choice for codes over large alphabets and there has been a recent resurgence of interest in MDS codes and locally recoverable codes (see [7] and [27]) due to their applications to distributed storage systems. There are further applications of arcs to quantum physics, see [8] and [25], to cryptography [21], to group theory [20] and to other combinatorial objects [51] and [63].

The articles we have referenced in these paragraphs are by no means exhaustive and are just given as an example of the far reaching relevance of arcs of finite projective spaces.

2. BASIC OBJECTS AND DEFINITIONS

A point of $\text{PG}(k-1, \mathbb{F})$ is a one-dimensional subspace of \mathbb{F}^k , and its *coordinate vector* with respect to a basis, is determined up to a nonzero scalar and is denoted by

$$(x_1, \dots, x_k).$$

The *weight* of a vector (or a point) with respect to some basis is the number of non-zero coordinates it has with respect to that basis.

An arc of $\text{PG}(k-1, \mathbb{F})$ of size $k+1$ is called a *frame* of $\text{PG}(k-1, \mathbb{F})$. An arc is called *complete* if it is not contained in a larger arc. An arc of size $\leq k$ in $\text{PG}(k-1, \mathbb{F})$ is always incomplete. Often we will implicitly assume that an arc has size at least $k+1$.

Let $\text{PGL}(k, \mathbb{F})$ denote the projective linear group. Two sets of points A and B in $\text{PG}(k-1, \mathbb{F})$ are called *projectively equivalent* if there exists a *projectivity* $\varphi \in \text{PGL}(k, \mathbb{F})$ such that $B = A^\varphi$.

A symmetric bilinear form $b(X, Y)$ on \mathbb{F}^k (where X and Y , stand for X_1, \dots, X_k and Y_1, \dots, Y_k) is *degenerate* if $b(X, y) = 0$ for some point y . A quadratic form $f(X)$ is *degenerate* if $f(y) = 0$ and $b(X, y) = 0$ for some point y , where

$$b(X, Y) = f(X + Y) - f(X) - f(Y)$$

is the bilinear form associated to $f(X)$. The zero-set in $\text{PG}(2, \mathbb{F})$ of a (non-degenerate) quadratic form on \mathbb{F}^3 is called a (*non-degenerate*) *conic*. We note that there is a unique conic through an arc of size 5 in $\text{PG}(2, \mathbb{F})$.

3. NORMAL RATIONAL CURVES

A *normal rational curve* is a set of points in $\text{PG}(k-1, \mathbb{F})$, projectively equivalent to

$$(1) \quad \mathcal{N}_{k-1} = \{(1, t, \dots, t^{k-1}) \mid t \in \mathbb{F}\} \cup \{(0, \dots, 0, 1)\}.$$

Lemma 1. *A normal rational curve in $\text{PG}(k-1, \mathbb{F})$, with $|\mathbb{F}| \geq k-1$, is an arc of $\text{PG}(k-1, \mathbb{F})$.*

Proof. The condition that k points of the normal rational curve as defined in (1) are linearly independent is equivalent to a Vandermonde determinant being different from zero. \square

Remark 2. *If the condition $|\mathbb{F}| \geq k-1$ is not satisfied, say $\mathbb{F} = \mathbb{F}_q$ with $q < k-1$, then the normal rational curve \mathcal{N}_{k-1} has $q+1 < k$ points. Therefore \mathcal{N}_{k-1} spans at most a hyperplane of $\text{PG}(k-1, \mathbb{F})$. In this case $t^{k-2} = t^i$ for some $1 \leq i < k-2$ for all $t \in \mathbb{F}$ and \mathcal{N}_{k-1} defines a normal rational curve in the subspace spanned by \mathcal{N}_{k-1} .*

For $k = 3$, a normal rational curve is the zero-set of a quadratic form, i.e. it is a conic. In the example (1) above, the quadratic form is $X_1X_3 - X_2^2$.

Since an arc of size $k + 1$ is a *frame* of $\text{PG}(k - 1, \mathbb{F})$ and $\text{PGL}(k, \mathbb{F})$ acts regularly on the set of frames of $\text{PG}(k - 1, \mathbb{F})$, any two arcs of size $k + 1$ are projectively equivalent.

The normal rational curve as defined in (1) is the image in $\text{PG}(k - 1, \mathbb{F})$ of the set of points of $\text{PG}(1, \mathbb{F})$ under the polynomial map

$$(2) \quad \nu_{k-1} : (x_1, x_2) \mapsto (x_1^{k-1}, x_1^{k-2}x_2, \dots, x_2^{k-1}).$$

The map ν_{k-1} can be used to define an action of the group $\text{PGL}(2, \mathbb{F})$ on $\text{PG}(k - 1, \mathbb{F})$ as follows.

Lemma 3. *Given $\alpha \in \text{PGL}(2, \mathbb{F})$, there is a projectivity $\tilde{\alpha} \in \text{PGL}(k, \mathbb{F})$ satisfying $\nu_{k-1}(x)^{\tilde{\alpha}} = \nu_{k-1}(x^\alpha)$ for all points x of $\text{PG}(1, \mathbb{F})$. Moreover, if $|\mathbb{F}| \geq k$, then $\tilde{\alpha}$ is unique.*

Proof. It is straightforward to verify that for each $\alpha \in \text{PGL}_2(\mathbb{F})$, the map $\nu_{k-1}(x) \mapsto \nu_{k-1}(x^\alpha)$ defines a projectivity $\tilde{\alpha}$ of $\text{PG}(k - 1, \mathbb{F})$. The uniqueness follows from the fact that if $|\mathbb{F}| \geq k$ then the image of ν_{k-1} contains a frame and a projectivity is uniquely determined by its action on a frame. \square

The action of the group $\text{PGL}(2, \mathbb{F})$ on $\text{PG}(k - 1, \mathbb{F})$ is then defined by lifting the action on $\text{PG}(1, \mathbb{F})$:

$$(3) \quad \tilde{\varphi} : \text{PGL}(2, \mathbb{F}) \rightarrow \text{PGL}(k, \mathbb{F}) : \alpha \mapsto \tilde{\alpha},$$

where $\tilde{\alpha}$ is the projectivity induced by α as in the proof of Lemma 3.

Lemma 4. *The image of $\tilde{\varphi}$ is a subgroup $H \cong \text{PGL}(2, \mathbb{F})$ of $\text{PGL}(k - 1, \mathbb{F})$ and the action defined by the restriction of H to the points of the normal rational curve \mathcal{N}_{k-1} from (1) is faithful. In particular, H acts regularly on the set of triples of distinct points of \mathcal{N}_{k-1} .*

Proof. The map $\tilde{\varphi}$ defined in (3) is a group homomorphism with trivial kernel. \square

Lemma 5. *If $|\mathbb{F}| \geq k + 1$ then there is a unique normal rational curve through an arc in $\text{PG}(k - 1, \mathbb{F})$ of size $k + 2$.*

Proof. Consider an arc in $\text{PG}(k - 1, \mathbb{F})$ consisting of the $k + 2$ points p_0, \dots, p_{k+1} . Suppose $p_0 = (u_1, \dots, u_k)$ and $p_{k+1} = (v_1, \dots, v_k)$ with respect to the basis p_1, \dots, p_k . Observe that this implies that the vectors (u_i, v_i) , $i = 1, \dots, k$, define k distinct points in $\text{PG}(1, \mathbb{F})$. Now consider the polynomial map

$$\varphi_f : (x_1, x_2) \mapsto (f_1(x_1, x_2), \dots, f_k(x_1, x_2))$$

from $\text{PG}(1, \mathbb{F})$ to $\text{PG}(k - 1, \mathbb{F})$, where

$$f_i(X_1, X_2) = \prod_{j=1, j \neq i}^k (u_j^{-1}X_1 - v_j^{-1}X_2), \quad i = 1, \dots, k$$

are forms of degree $k - 1$. Since $\sum a_i f_i = 0$ implies $a_i = 0$ after substituting (u_i, v_i) for (X_1, X_2) , the forms f_1, \dots, f_k form a basis for the space of forms of degree $k - 1$ in $\mathbb{F}[X_1, X_2]$. This implies that up to a change of basis the image of φ_f coincides with that of ν_{k-1} as in (4), and is therefore a normal rational curve \mathcal{N}_f in $\text{PG}(k - 1, \mathbb{F})$. The image under φ_f of the point (u_l, v_l) , for $l \in \{1, \dots, k\}$, is the point p_l , since $f_i(u_l, v_l) = 0$ for $i \neq l$ and $f_i(u_l, v_l) = \prod_{j \neq i} (u_j^{-1}u_l - v_j^{-1}v_l)$, which is nonzero. The image under φ_f of the points $(1, 0)$ and $(0, 1)$ are the points p_0 and p_{k+1} . This shows that the points p_0, \dots, p_{k+1} are contained in a normal rational curve in $\text{PG}(k - 1, \mathbb{F})$.

The uniqueness follows by considering any normal rational curve \mathcal{N}_g given by the image of

$$\varphi_g : (x_1, x_2) \mapsto (g_1(x_1, x_2), \dots, g_k(x_1, x_2))$$

where the g_i 's form a basis for the space of forms of degree $k - 1$ in $\mathbb{F}[X_1, X_2]$, and assuming that \mathcal{N}_g contains the $k + 2$ points p_0, \dots, p_{k+1} . W.l.o.g. we may assume that the points $(1, 0)$ and $(0, 1)$ are

the pre-images of the points p_0 and p_{k+1} (if necessary, apply a coordinate transformation to $\text{PG}(1, \mathbb{F})$ and relabel the g_i 's). For each $i \in \{1, \dots, k\}$, the assumption $p_i \in \mathcal{N}_g$ implies the existence of a point (a_i, b_i) in $\text{PG}(1, \mathbb{F})$ and a factor $a_i^{-1}X_1 - b_i^{-1}X_2$ of each g_j , $j \neq i$. This determines each of the g_i 's up to a nonzero scalar factor. The condition that $\varphi_g(1, 0) = p_0$ and $\varphi_g(0, 1) = p_{k+1}$ implies that $(u_1, \dots, u_k) = \lambda(a_1, \dots, a_k)$ and $(v_1, \dots, v_k) = \mu(b_1, \dots, b_k)$ and therefore

$$g_i(X_1, X_2) = \prod_{j=1, j \neq i}^k (u_j^{-1}\lambda X_1 - v_j^{-1}\mu X_2), \quad i = 1, \dots, k.$$

It follows that $\mathcal{N}_f = \mathcal{N}_g$, since $\mathcal{N}_g = \mathcal{N}_g^{\alpha}$ where α is the projectivity of $\text{PG}(1, \mathbb{F})$ mapping (x_1, x_2) to $(\lambda x_1, \mu x_2)$ (see Lemma 3 and Lemma 4). \square

Lemma 6. *The projection $x(\mathcal{N})$ of a normal rational curve \mathcal{N} in $\text{PG}(k-1, \mathbb{F})$ from one of its points x onto a hyperplane Π , $x \notin \Pi$, is contained in a normal rational curve of Π , which is uniquely determined by $x(\mathcal{N})$ if $|\mathbb{F}| \geq k+1$.*

Proof. Let \mathcal{A} be the set of points on the normal rational curve as defined in (1). The projection from the point $(1, 0, \dots, 0)$ onto the hyperplane Π with equation $X_1 = 0$ is the set of points $\{(0, 1, t, \dots, t^{k-2}) \mid t \in \mathbb{F} \setminus \{0\}\} \cup \{(0, \dots, 0, 1)\}$, to which we can add the point $(0, 1, 0, \dots, 0)$ to obtain a normal rational curve in Π . Applying Lemma 5 and the transitivity properties from Lemma 4 concludes the proof. \square

A normal rational curve is a nonsingular algebraic curve in $\text{PG}(k-1, \mathbb{F})$ and the *tangent line* at the point $p = \nu_{k-1}(x_0, x_1)$ of \mathcal{N}_{k-1} is the line $\ell_p = \langle p, p_0, p_1 \rangle$ where p_i is the point whose coordinates are the evaluations at (x_0, x_1) of the partial derivatives of the coordinates of $\nu_{k-1}(X_0, X_1)$ with respect to X_i ($i = 0, 1$).

Lemma 7. *The tangents to a normal rational curve \mathcal{N}_k , with $k \geq 4$, are pairwise disjoint and only meet the secants of \mathcal{N}_k which pass through the points of tangency.*

Proof. Observing that the tangent lines at the points $\nu_{k-1}(1, 0)$ and $\nu_{k-1}(0, 1)$ are disjoint, the first part of the lemma follows from the transitivity properties from Lemma 4. For the second part, consider the tangent line ℓ_p at the point $\nu_{k-1}(1, 0)$ and the secant line ℓ through the points $\nu_{k-1}(0, 1)$ and $\nu_{k-1}(1, 1)$. Observing that the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

has rank 4, implies that the two lines ℓ_p and ℓ are disjoint. \square

Lemma 8. *If x is a point of a normal rational curve $\mathcal{N} = \mathcal{N}_{k-1}$ in $\text{PG}(k-1, \mathbb{F})$ with $|\mathbb{F}| \geq k+1$, Π a hyperplane not passing through x , and \mathcal{N}_x is the unique normal rational curve passing through the projection $x(\mathcal{N})$ of \mathcal{N} from x onto Π , then the tangent line of \mathcal{N} at x is the line through x and the unique point of $\mathcal{N}_x \setminus x(\mathcal{N})$.*

Proof. By the transitivity properties from Lemma 4 we may assume $x = \nu_{k-1}(1, 0)$. The result then immediately follows from the details given in the proofs of Lemma 6 and Lemma 7. \square

4. EXAMPLES OF LARGE ARCS IN $\text{PG}(k-1, q)$

From now on we consider the case $\mathbb{F} = \mathbb{F}_q$ for some prime power q . As we have seen in the previous section, when $q \geq k-1$, a normal rational curve is an example of an arc of size $q+1$ in $\text{PG}(k-1, q)$. If $k = 3$ then a normal rational curve is a non-degenerate conic \mathcal{C} , and if q is even, then the tangents

to \mathcal{C} are concurrent. The point N common to all the tangents is called the *nucleus* of \mathcal{C} . It follows that $\mathcal{C} \cup \{N\}$ is also an arc. This gives an example of what is called a *hyperoval*: an arc of size $q + 2$ in $\text{PG}(2, q)$. So a hyperoval can be obtained from a non-degenerate conic \mathcal{C} in $\text{PG}(2, q)$, q even, by adding the nucleus of \mathcal{C} . Another example of a hyperoval is the following.

Example 9 (Segre [54]). Let σ be the automorphism of \mathbb{F}_q , $q = 2^h$, which takes x to x^{2^e} . The set

$$\mathcal{A} = \{(1, t, t^\sigma) \mid t \in \mathbb{F}_q\} \cup \{(0, 0, 1), (0, 1, 0)\}$$

is an arc of $q + 2$ points in $\text{PG}(2, q)$, whenever $(e, h) = 1$. It is an example of a *translation hyperoval*.

Proof. We prove that \mathcal{A} defined in Example 9 is an arc. Clearly, the line through $(0, 0, 1)$ and $(0, 1, 0)$ ($X_1 = 0$), as well as the lines through one of $\{(0, 0, 1), (0, 1, 0)\}$ and a point $(1, t, t^\sigma)$ ($tX_1 = X_2$ and $t^\sigma X_1 = X_3$) contains no other points of \mathcal{A} . If three distinct points (parameterised by t_1, t_2, t_3) of \mathcal{A} were collinear then, by calculating the corresponding determinant, one obtains

$$\left(\frac{t_3 - t_1}{t_2 - t_1}\right)^{\sigma-1} = 1.$$

Since $(e, h) = 1$ this implies $t_2 = t_3$, a contradiction. \square

We refer to [50], [48] and the more recent [60, Section 1] and to Section 15.1 in this article, for more on hyperovals.

Next we give two further examples of large arcs (of size $q + 1$) which are not normal rational curves. The first example is an example in $\text{PG}(3, q)$, q even, the second in $\text{PG}(4, q)$, $q = 9$. As we will see, such examples are extremely rare.

Example 10 (Segre [57]). Let σ be the automorphism of \mathbb{F}_q , $q = 2^h$, which takes x to x^{2^e} . The set

$$\mathcal{A} = \{(1, t, t^\sigma, t^{\sigma+1}) \mid t \in \mathbb{F}_q\} \cup \{(0, 0, 0, 1)\}$$

is an arc of $q + 1$ points in $\text{PG}(3, q)$, whenever $(e, h) = 1$. Note that \mathcal{A} is a normal rational curve when $e = 1$.

Proof. Consider the map

$$(4) \quad \varphi_1 : (x_1, x_2) \mapsto (x_1^{\sigma+1}, x_1^\sigma x_2, x_1 x_2^\sigma, x_2^{\sigma+1})$$

from $\text{PG}(1, q)$ to $\text{PG}(3, q)$. Then \mathcal{A} is the image of φ_1 , and lifting the action of $\text{PGL}(2, q)$ on $\text{PG}(1, q)$ to $\text{PG}(3, q)$, (in the same way as in (3), but now with φ_1 instead of φ) we obtain the action of a subgroup $H \leq \text{PGL}_4(\mathbb{F}_q)$, $H \cong \text{PGL}_2(q)$, on the set of points of \mathcal{A} , which is regular on the set of triples of distinct points of \mathcal{A} . Now consider any four distinct points of \mathcal{A} . Using the action of H , we may assume that the first three points are the images under φ_1 of the points $(1, 0)$, $(0, 1)$ and $(1, 1)$. Let $(1, t, t^\sigma, t^{\sigma+1})$, $t \in \mathbb{F}_q \setminus \{0, 1\}$ be the fourth point. Then these four points are collinear if and only if $t^\sigma = t$, a contradiction since $t \notin \{0, 1\}$ and $t^\sigma = t^{2^e}$ with $(e, h) = 1$. \square

Example 11 (Glynn [23]). Let η be an element of \mathbb{F}_9 , $\eta^4 = -1$. The set

$$\mathcal{A} = \{(1, t, t^2 + \eta t^6, t^3, t^4) \mid t \in \mathbb{F}_9\} \cup \{(0, 0, 0, 0, 1)\}.$$

is an arc of size $q + 1$ in $\text{PG}(4, 9)$.

Proof. Consider five distinct points of \mathcal{A} (for the moment excluding the point $(0, 0, 0, 0, 1)$) parameterised by t_1, \dots, t_5 . Let A denote the 5×5 -matrix, with the corresponding vectors as rows. The determinant of A equals $|A_1| + \eta|A_2|$, where A_1 and A_2 are equal to A , except in their third column, where A_1 has entries t_i^2 , and A_2 has ηt_i^6 , instead of the entries $t_i^2 + \eta t_i^6$ in A . If $|A| = 0$ then $|A_1| = -\eta|A_2|$. Applying the automorphism $\sigma : a \mapsto a^3$ to the entries of A_1 gives the matrix obtained from A_2 after applying the permutation (24) to its columns. This implies that $|A_1|^3 = -|A_2|$ and so $|A_1|^4 = |A_2|^4$, since $|A_1|^{12} = |A_1|^4$. It follows from $|A_1| = -\eta|A_2|$ that $|A_1|^4 = \eta^4|A_1|^4$. Since A_1 is a

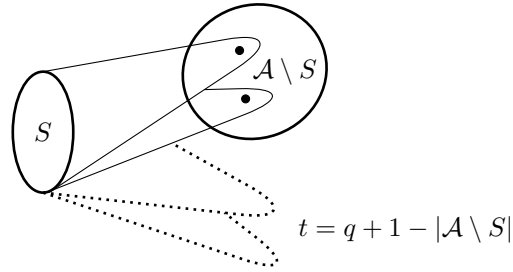
Vandermonde matrix, $|A_1| \neq 0$, which contradicts $\eta^4 = -1$.

If one of the five points is the point $(0, 0, 0, 0, 1)$, then the same arguments work, starting from the 4×4 matrix A whose rows correspond to the vectors of \mathcal{A} parametrised by 4 distinct values of t , but with the last coordinate removed. \square

5. THE TRIVIAL UPPER BOUND AND THE MDS CONJECTURE

In this section we give the trivial upper bounds on the size of an arc, explain the links between arcs and MDS codes and conclude with the statement of the main conjecture for MDS codes.

Lemma 12. *If S is a $(k-2)$ -subset of an arc \mathcal{A} in $\text{PG}(k-1, q)$ then there are exactly $t = q + k - |\mathcal{A}| - 1$ hyperplanes which meet \mathcal{A} in precisely S .*



Proof. There are $q + 1$ hyperplanes containing the $(k - 3)$ -dimensional subspace spanned by S , each containing at most one point of $\mathcal{A} \setminus S$. Therefore there are $q + 1 - |\mathcal{A} \setminus S|$ hyperplanes which meet \mathcal{A} in precisely the points of S . \square

Theorem 13. *An arc of $\text{PG}(k - 1, q)$ has at most $q + k - 1$ points.*

Proof. This follows from Lemma 12, since $t \geq 0$. \square

Theorem 14. *Let \mathcal{A} be an arc of $\text{PG}(k - 1, q)$. If $k \geq q$ then $|\mathcal{A}| \leq k + 1$.*

Proof. After choosing a suitable basis we may assume that \mathcal{A} contains the standard frame

$$\{e_1, \dots, e_k, e_1 + \dots + e_k\},$$

where e_i is the i -th vector in the standard basis of \mathbb{F}_q^k . Suppose $u = (u_1, \dots, u_k) \in \mathcal{A} \setminus \{e_1, \dots, e_k, e_1 + \dots + e_k\}$. If $u_i = 0$ for some i then the hyperplane with equation $X_i = 0$ contains k points of \mathcal{A} , contradicting the arc property. If $u_i \neq 0$ for all i then, since $k \geq q$, by the pigeon-hole principle there exists i and j such that $u_i = u_j$. But then the hyperplane with equation $X_i = X_j$, contains k points of \mathcal{A} , contradicting the arc property. \square

Let $G_{\mathcal{A}}$ denote the $k \times |\mathcal{A}|$ matrix whose columns are the coordinate vectors of the points belonging to a set \mathcal{A} in $\text{PG}(k - 1, q)$.

Lemma 15. *The set \mathcal{A} is an arc in $\text{PG}(k - 1, q)$ if and only if for all non-zero $u \in \mathbb{F}_q^k$ the vector $uG_{\mathcal{A}}$ has at most $k - 1$ zeros.*

Proof. The i -th coordinate of $uG_{\mathcal{A}}$ is zero if and only if the point of \mathcal{A} corresponding to the i -th column of $G_{\mathcal{A}}$ lies in the hyperplane defined by u . \square

Lemma 16. *A non-zero vector in $\{uG_{\mathcal{A}} \mid u \in \mathbb{F}_q^k\}$ has weight at least $|\mathcal{A}| - k + 1$.*

Proof. This follows immediately from Lemma 15. \square

A k -dimensional *linear code* of length n and minimum distance d is a k -dimensional subspace of \mathbb{F}_q^n in which every non-zero vector has weight at least d . Such a code is called an \mathbb{F}_q -*linear* $[n, k, d]$ -*code*, where n , k , and d are often referred to as the *parameters* of the code.

Let C be a linear code of length n and minimum distance d . If \bar{u} denotes the vector obtained from the codeword $u \in C$ by deleting the first $d - 1$ coordinates from u , then clearly for any two codewords $u \neq v$, we must have $\bar{u} \neq \bar{v}$. This implies that the dimension k of C can be at most $n - (d - 1)$. Equivalently $d \leq n - k + 1$. This bound is called the *Singleton bound*. A k -dimensional *linear maximum distance separable* (MDS) code C of length n is a k -dimensional subspace of \mathbb{F}_q^n in which every non-zero vector has weight at least $n - k + 1$. We have already established the following theorem.

Theorem 17. *The linear code C generated by the matrix G , whose columns are the coordinate vectors of the points of an arc is a linear MDS code, and vice versa, the set of columns of a generator matrix of an MDS code considered as a set of points of the projective space, is an arc.*

The *dual* of a linear code C of length n over \mathbb{F}_q is,

$$C^\perp = \{v \in \mathbb{F}_q^n \mid u \cdot v = 0 \text{ for all } u \in C\},$$

where $u \cdot v$ denotes the standard inner product $u \cdot v = u_1v_1 + \dots + u_nv_n$. If C has dimension k , then C^\perp has dimension $n - k$.

Lemma 18. *The linear code C is MDS if and only if C^\perp is MDS.*

Proof. Suppose C is a k -dimensional MDS code of length n and that C^\perp is not MDS. Then C^\perp contains a non-zero vector v of weight less than $n - (n - k) + 1 = k + 1$. Let G be the generator matrix of C . Since $v \in C^\perp$ if and only if $Gv = 0$, the (at most k) non-zero coordinates of v give a non-trivial linear dependence relation between at most k columns of $G_{\mathcal{A}}$. This contradicts the fact that the columns of G form an arc in $\text{PG}(k - 1, q)$. \square

Corollary 19. *There is an arc of size n in $\text{PG}(k - 1, q)$ if and only if there is an arc of size n in $\text{PG}(n - k - 1, q)$.*

Proof. This follows from Theorem 17 and Lemma 18. \square

We have seen examples (normal rational curves and some others) of arcs of size $q + 1$ in $\text{PG}(k - 1, q)$ (for $q \geq k - 1$) and of size $q + 2$ in $\text{PG}(2, q)$.

If \mathcal{A} is the set of points on the normal rational curve as defined in (1), and $G_{\mathcal{A}}$ is the generator matrix of the associated MDS code, as described above, then for each $u = (u_0, \dots, u_{k-1}) \in \mathbb{F}_q^k$, we obtain a codeword $c = uG_{\mathcal{A}}$, whose last coordinate is u_{k-1} and whose other coordinates are of the form

$$u_0 + u_1t + u_2t^2 + \dots + u_{k-1}t^{k-1}, \quad t \in \mathbb{F}_q.$$

Therefore, the code $C = \{uG_{\mathcal{A}} \mid u \in \mathbb{F}_q^k\}$ can be rewritten as

$$(5) \quad C = \{(f(t_1), f(t_2), \dots, f(t_q), f_{k-1}) \mid f(X) = \sum_{i=0}^{k-1} f_i X^i \in \mathbb{F}_q[X]\}$$

where we have labeled the q elements of \mathbb{F}_q as $\{t_1, t_2, \dots, t_q\}$. This MDS code obtained from a normal rational curve is called the *Reed-Solomon code* (or *extended Reed-Solomon code*) and has parameters $[q + 1, k, q - k + 2]$.

Theorem 20. *The dual of the Reed-Solomon code is a Reed-Solomon code.*

Proof. Let C_k denote the Reed-Solomon code obtained from the normal rational curve in $\text{PG}(k-1, q)$ as in (5). Let $c_f \in C_k$ and $c_g \in C_{q+1-k}$ with $f, g \in \mathbb{F}_q[X]$. The inner product of c_f and c_g is

$$\sum_{t \in \mathbb{F}_q} f(t)g(t) + f_{k-1}g_{q-k}.$$

Since the sum of all n -th powers $\{t^n : t \in \mathbb{F}_q\}$ is zero unless n is a multiple of $q-1$ in which case it is -1 , the above sum is zero. \square

By Corollary 19 the normal rational curves also give arcs of size $q+1$ in $\text{PG}(q-k, q)$ and the hyperovals give arcs of size $q+2$ in $\text{PG}(q-3, q)$. If we exclude the trivial parameters, so assume $3 \leq k \leq q-2$, and we also exclude the case of hyperovals (including the arcs obtained from the dual codes of a hyperoval code), we obtain the interval $4 \leq k \leq q-3$ for the dimension k of an \mathbb{F}_q -linear MDS code. For these dimensions, no \mathbb{F}_q -linear MDS codes are known which have length larger than $q+1$. Equivalently, if $4 \leq k \leq q-3$, then no arcs in $\text{PG}(k-1, q)$ are known of size larger than $q+1$. This led to the well-known *MDS-conjecture*, see for example [44, Chapter 11].

The MDS conjecture can be extended to non-linear codes but that was not done until recently, see [36]. For the purposes of this article, the MDS conjecture will refer to the MDS conjecture for linear codes.

Conjecture 21. (*The MDS conjecture*) *If $4 \leq k \leq q-3$ then an arc in $\text{PG}(k-1, q)$ has size at most $q+1$.*

In fact, although we have already seen some examples (precisely Examples 10, 11) of arcs which have the same size as but are not equal to normal rational curves, it is our belief that all arcs of size $q+1$ are already known, excluding the cases $k \in \{3, q-1\}$ and q is even.

Conjecture 22. *If $6 \leq k \leq q-5$ then an arc in $\text{PG}(k-1, q)$ of size $q+1$ is a normal rational curve.*

Observe that Theorem 20 implies the following.

Theorem 23. *For $k \geq 3$ and $q-k \geq 3$, if every arc of size $q+1$ in $\text{PG}(k-1, q)$ is a normal rational curve then so is every arc of size $q+1$ in $\text{PG}(q-k, q)$.*

In the next sections we will give a proof for several instances of the MDS conjecture. In Section 19, we summarize the state of the art on the MDS conjecture.

6. THE PROJECTION THEOREM

Most results on arcs are, or rely on, results on planar arcs, i.e. arcs in $\text{PG}(2, q)$. The reason for this, besides the fact that it is the smallest interesting case, is the *projection theorem* (Theorem 24), which asserts that if two distinct projections of an arc are contained in a normal rational curve then so is the arc. The fundamental ideas were developed by Segre in [53] in which the MDS conjecture is proven for $k \in \{4, 5\}$ and q odd. The papers of Beniamino Segre published in the 1950's and 1960's had a huge influence on the development of combinatorial and geometric techniques which were used (first by himself and later by others) to prove several instances of the MDS conjecture. The proof of his celebrated theorem which says that an arc of size $q+1$ in $\text{PG}(2, q)$, q odd, is a conic (compare to Conjecture 22) is just one example. Segre's work was so influential that it can hardly be overestimated, and mathematics today still benefits from his efforts.

Theorem 24 (Projection theorem). *Let \mathcal{A} be an arc in $\text{PG}(k-1, q)$, with $k \geq 4$ and $q \geq k+1$. If \mathcal{A} projects onto a subset of a normal rational curve in a hyperplane of $\text{PG}(k-1, q)$, for two distinct points of \mathcal{A} , then \mathcal{A} is contained in a normal rational curve.*

Proof. We may assume that \mathcal{A} has size at least $k + 3$, since by Lemma 5 every arc of size $\leq k + 2$ is contained in a normal rational curve. Suppose that each of the projections $x(\mathcal{A})$ and $y(\mathcal{A})$ from the points $x, y \in \mathcal{A}$, $x \neq y$, is contained in a normal rational curve, say \mathcal{N}_x and \mathcal{N}_y . Consider any subset S of \mathcal{A} of size $k + 2$, containing x, y , and let \mathcal{N} be the unique normal rational curve containing S (here we use $q \geq k + 1$). By the hypothesis, Lemma 5 and Lemma 6, both the arc \mathcal{A} and the normal rational curve \mathcal{N} are then contained in the two distinct cones $x\mathcal{N}_x$ and $y\mathcal{N}_y$. Consider a point $z \in \mathcal{A} \setminus \{x, y\}$, and let $\ell_x = xz$ and $\ell_y = yz$. Suppose ℓ_x does not contain a point of \mathcal{N} . Then ℓ_x is the tangent line of \mathcal{N} at x , since $x(\ell_x)$ is then the unique point of $\mathcal{N}_x \setminus x(\mathcal{N})$ (see Lemma 8). Also ℓ_y is either a secant or the tangent line of \mathcal{N} at y . But then the fact that ℓ_x and ℓ_y meet contradicts Lemma 7. Therefore neither of the lines ℓ_x and ℓ_y are tangents. But then $z \in \mathcal{N}$ since the plane $\langle \ell_x, \ell_y \rangle$ can contain at most 3 points of \mathcal{N} . This shows that \mathcal{A} is contained in \mathcal{N} . \square

Corollary 25. *Let \mathcal{A} be an arc in $\text{PG}(k - 1, q)$, with $k \geq 4$ and $q \geq k + 1$. If the projection of \mathcal{A} from each $(k - 3)$ -subset of a $(k - 2)$ -subset of \mathcal{A} is contained in a conic then \mathcal{A} is contained in a normal rational curve.*

Proof. The proof follows by induction on k from Theorem 24. It coincides with Theorem 24 in the case $k = 4$. The hypothesis for $k = r$ implies the hypothesis for a projected arc (from one of the points in the $(k - 2)$ -subset) for $k = r - 1$, from which one can apply the induction hypothesis and then Theorem 24. \square

Corollary 26. *If every arc of size $q + 1$ in $\text{PG}(k - 1, q)$ with $q + 1 \geq k + 3 \geq 6$ is a normal rational curve then the MDS conjecture holds true in $\text{PG}(k, q)$.*

Proof. Suppose \mathcal{A} is an arc in $\text{PG}(k, q)$ of size $q + 2$. Projecting \mathcal{A} from any of its points gives an arc of size $q + 1$ in $\text{PG}(k - 1, q)$ which by hypothesis is a normal rational curve. By Theorem 24, \mathcal{A} is contained in a normal rational curve, which is a contradiction since a normal rational curve has $q + 1$ points. \square

7. THE LEMMA OF TANGENTS

Let \mathcal{A} be an arc of $\text{PG}(k - 1, q)$ of size $q + k - 1 - t$ in which we arbitrarily order the points of \mathcal{A} . Let S be a subset of \mathcal{A} of size $k - 2$ ordered as in \mathcal{A} . Let $\alpha_1, \dots, \alpha_t$ be t linear forms whose kernels are the t hyperplanes which meet \mathcal{A} in precisely S , see Lemma 12. Define, up to a scalar factor, a homogeneous polynomial of degree t ,

$$(6) \quad f_S(X) = \prod_{i=1}^t \alpha_i(X),$$

where $X = (X_1, \dots, X_k)$. The zero locus $\mathcal{Z}(f_S)$ of f_S in $\text{PG}(k - 1, q)$ is called the *tangent hypersurface of \mathcal{A} at S* . For any permutation σ of the elements of S , define

$$f_{\sigma(S)}(X) = (-1)^{s(\sigma)(t+1)} f_S(X),$$

where $s(\sigma)$ is the parity of the permutation σ .

A homogeneous polynomial f in k variables defines a function \tilde{f} from \mathbb{F}_q^k to \mathbb{F}_q under evaluation. If we change the basis of \mathbb{F}_q^k then although the polynomial f will change, the function \tilde{f} will not. Put another way, any function from \mathbb{F}_q^k to \mathbb{F}_q is the evaluation of a polynomial once we fix a basis of \mathbb{F}_q^k . Obviously, the polynomial we obtain depends on the basis we choose.

The following is the coordinate-free version of Segre's lemma of tangents.

Lemma 27 (Coordinate-free lemma of tangents). *Let \mathcal{A} be an arc of $\text{PG}(k - 1, q)$ and let D be a subset of \mathcal{A} of size $k - 3$. For all $x, y, z \in \mathcal{A} \setminus D$,*

$$f_{D \cup \{x\}}(y) f_{D \cup \{y\}}(z) f_{D \cup \{z\}}(x)$$

$$= (-1)^{t+1} f_{D \cup \{y\}}(x) f_{D \cup \{z\}}(y) f_{D \cup \{x\}}(z).$$

Proof. Let f_a^* be the polynomial we obtain from $f_{D \cup \{a\}}$ with respect to the basis $B = \{x, y, z\} \cup D$.

Then

$$f_x^*(X) = \prod_{i=1}^t (a_{i2}X_2 + a_{i3}X_3), \quad f_y^*(X) = \prod_{i=1}^t (b_{i1}X_1 + b_{i3}X_3), \quad f_z^*(X) = \prod_{i=1}^t (c_{i1}X_1 + c_{i2}X_2),$$

for some $a_{ij}, b_{ij}, c_{ij} \in \mathbb{F}_q$. Let $s \in \mathcal{A} \setminus B$. The hyperplane joining s and $D \cup \{x\}$ is $\ker(s_3X_2 - s_2X_3)$ where (s_1, s_2, \dots, s_k) are the coordinates of s with respect to the basis B .

As s runs through the elements of $\mathcal{A} \setminus B$, the element $-s_2/s_3$ runs through the elements of $\mathbb{F}_q \setminus \{a_{i3}/a_{i2} \mid i = 1, \dots, t\}$, the latter being excluded since the tangent hypersurface of \mathcal{A} at $D \cup \{x\}$ does not pass through s . Using the fact that the product of all the non-zero elements of \mathbb{F}_q is -1 , we obtain

$$\prod_{s \in \mathcal{A} \setminus B} \frac{-s_2}{s_3} \prod_{i=1}^t \frac{a_{i3}}{a_{i2}} = -1,$$

and since $\prod_{i=1}^t a_{i3} = f_x^*(z)$ and $\prod_{i=1}^t a_{i2} = f_x^*(y)$, we have

$$f_x^*(z) \prod_{s \in \mathcal{A} \setminus B} (-s_2) = f_x^*(y) \prod_{s \in \mathcal{A} \setminus B} s_3.$$

This equation was obtained by considering all hyperplanes through $D \cup \{x\}$. Similarly, by considering the hyperplanes through $D \cup \{y\}$, we get

$$f_y^*(x) \prod_{s \in \mathcal{A} \setminus B} (-s_3) = f_y^*(z) \prod_{s \in \mathcal{A} \setminus B} s_1$$

and by considering the hyperplanes through $D \cup \{z\}$ we get

$$f_z^*(y) \prod_{s \in \mathcal{A} \setminus B} (-s_1) = f_z^*(x) \prod_{s \in \mathcal{A} \setminus B} s_2.$$

Combining these three equations, we obtain

$$f_x^*(z) f_y^*(x) f_z^*(y) = (-1)^{t+1} f_x^*(y) f_y^*(z) f_z^*(x).$$

Now, since f^* and f define the same functions on the points of $\text{PG}(k-1, q)$, the lemma follows. \square

Let \mathcal{E} be the set of the first $k-2$ elements of \mathcal{A} . For each $(k-2)$ -subset $S \subset \mathcal{A}$, scale the polynomial $f_S(X)$ so that

$$(7) \quad f_S(e) = (-1)^{s(t+1)} f_{S \cup \{e\} \setminus \{a\}}(a),$$

where e is the first element of $\mathcal{E} \setminus S$, a is the last element of $S \setminus \mathcal{E}$, and s is the parity of the permutation which orders $S \cup \{e\}$ as in the ordering of \mathcal{A} (to determine the value of s we assume the ordering of \mathcal{A} for the subset S). With this notation it should be understood that the order is respected when taking the union of ordered sets, i.e. with “union” we mean the concatenation of the ordered sets.

So that this definition makes sense, we fix a vector representative for each point of \mathcal{A} . This is essential, since the scaling will depend on which vector representative we choose for a particular point.

Lemma 28. *Let \mathcal{A} be an arc of $\text{PG}(k-1, q)$ and let D be a subset of \mathcal{A} of size $k-3$. For all $x, y, z \in \mathcal{A} \setminus D$,*

$$f_{D \cup \{x\}}(y) = (-1)^{s(\sigma)(t+1)} f_{D \cup \{y\}}(x),$$

where σ is the permutation that orders $(D \cup \{x\}, y)$ as $(D \cup \{y\}, x)$ and $s(\sigma)$ is the parity of σ .

Proof. We will prove this inductively on $|D \cap \mathcal{E}|$. Suppose that $|D \cap \mathcal{E}| = k - 3$ and let e be the unique element of $\mathcal{E} \setminus D$. Applying Lemma 27 to D and x, y, e , we have that

$$f_{D \cup \{x\}}(y) f_{D \cup \{y\}}(e) f_{D \cup \{e\}}(x) = (-1)^{t+1} f_{D \cup \{y\}}(x) f_{D \cup \{e\}}(y) f_{D \cup \{x\}}(e).$$

Since we scaled $f_{D \cup \{x\}}(X)$ so that

$$f_{D \cup \{x\}}(e) = (-1)^{s(\theta)(t+1)} f_{D \cup \{e\}}(x),$$

where θ is the permutation that orders $(D \cup \{x\}, e)$ as $(D \cup \{e\}, x)$, and we scaled $f_{D \cup \{y\}}(X)$ so that

$$f_{D \cup \{y\}}(e) = (-1)^{s(\tau)(t+1)} f_{D \cup \{e\}}(y),$$

where τ is the permutation that orders $(D \cup \{y\}, e)$ as $(D \cup \{e\}, y)$, we have that

$$f_{D \cup \{x\}}(y) = (-1)^{(s(\theta)+s(\tau)+1)(t+1)} f_{D \cup \{y\}}(x).$$

It suffices to observe now that $s(\sigma) = s(\theta) + s(\tau) + 1$, where σ is the permutation that orders $(D \cup \{x\}, y)$ as $(D \cup \{y\}, x)$.

Now suppose that $|D \cap \mathcal{E}| \leq k - 4$ and that the statement holds for D such that $|D \cap \mathcal{E}|$ is larger. Let e be the first element of $\mathcal{E} \setminus (D \cup \{y\})$.

If $e = x$ then we are done since we scaled $f_{D \cup \{y\}}(X)$ so that

$$f_{D \cup \{y\}}(e) = (-1)^{s(\sigma)(t+1)} f_{D \cup \{e\}}(y),$$

where σ is the permutation that orders $(D \cup \{y\}, e)$ as $(D \cup \{e\}, y)$.

If $e \neq x$ then let z denote the last element of $D \cup \{y\}$. By scaling,

$$f_{D \cup \{y\}}(e) = (-1)^{s(\tau)(t+1)} f_{D \cup \{e, y\} \setminus \{z\}}(z),$$

where τ is the permutation that orders $(D \cup \{y\}, e)$ as $(D \cup \{e, y\} \setminus \{z\}, z)$. By induction,

$$f_{D \cup \{e, y\} \setminus \{z\}}(z) = (-1)^{s(\theta)(t+1)} f_{D \cup \{e\}}(y),$$

where $\theta \circ \tau$ is the permutation that orders $(D \cup \{y\}, e)$ as $(D \cup \{e\}, y)$.

Observe that e is also the first element of $\mathcal{E} \setminus (D \cup \{x\})$, so we also have that

$$f_{D \cup \{x\}}(e) = (-1)^{s(\alpha)(t+1)} f_{D \cup \{e\}}(x),$$

where α is the permutation that orders $(D \cup \{x\}, e)$ as $(D \cup \{e\}, x)$.

Now, applying Lemma 27, to D and x, y, e the lemma follows observing again that if σ is the permutation that orders $(D \cup \{y\}, x)$ as $(D \cup \{x\}, y)$ then

$$s(\sigma) = s(\theta) + s(\tau) + s(\alpha) + 1.$$

This concludes the proof. \square

Define the function g on ordered subsets of \mathcal{A} of size $k - 1$ by

$$(8) \quad g(S \cup \{a\}) = (-1)^{s(t+1)} f_S(a),$$

where S is an ordered subset of \mathcal{A} of size $k - 2$ and s is the parity of the permutation which orders $S \cup \{a\}$ as in the ordering of \mathcal{A} . Note that S is considered as an unordered set in the notation $f_S(a)$.

Lemma 29 (Scaled coordinate-free lemma of tangents). *If σ is a permutation in $\text{Sym}(k - 1)$ and C is an ordered $(k - 1)$ -subset of \mathcal{A} then*

$$g(C^\sigma) = (-1)^{s(t+1)} g(C),$$

where s is the parity of the permutation σ .

Proof. This follows from Lemma 28 and the observation that

$$(-1)^{s(\sigma\circ\tau)} = (-1)^{s(\sigma)+s(\tau)}$$

for permutations σ and τ . □

8. A SYSTEM OF EQUATIONS ASSOCIATED WITH AN ARC

Let \mathcal{A} be an arc of $\text{PG}(k-1, q)$ of size $q+k-1-t \geq k+t$ arbitrarily ordered and let E be a subset of \mathcal{A} of size $k+t$. For a $(k-1)$ -subset C of \mathcal{A} , we define $g(C)$ as in the previous section, see (8). We write

$$\det(u, C)$$

to mean the determinant of the matrix whose first row is u and whose remaining $k-1$ rows are the vectors in C in the order in which they are ordered in \mathcal{A} . Other expressions for \det are similarly defined and X is understood to mean the indeterminants (X_1, \dots, X_k) .

Lemma 30. *For any subset S of E of size $k-2$,*

$$\sum_C g(C) \prod_{u \in E \setminus C} \det(u, C)^{-1} = 0,$$

where the sum is over $(k-1)$ -subsets C of E containing S .

Proof. Let S be a subset of E of size $k-2$. Since there are $t+1$ determinants in the product, any permutation σ of C will change the sign by $(-1)^{s(\sigma)(t+1)}$ which, by Lemma 29, will be cancelled out by the same sign appearing due to the permutation of C in the $g(C)$ term. Thus, the summand in the equation is unaffected by the ordering of C . Hence, it suffices to prove the equation for ordered subsets C of E in which the first $k-2$ elements are S .

Let B be a subset of \mathcal{A} of size k containing S . With respect to the basis B , the polynomial $f_S(X)$ is a homogeneous polynomial in two variables of degree t .

Let x be a fixed element of $E \setminus S$. The polynomial

$$\sum_{e \in E \setminus (S \cup \{x\})} f_S(e) \prod_{u \in E \setminus (S \cup \{x, e\})} \frac{\det(X, u, S)}{\det(e, u, S)}$$

evaluates to $f_S(e)$ for all $e \in E \setminus (S \cup \{x\})$. Since this polynomial and $f_S(X)$ are both homogeneous polynomials in two variables of degree t which agree at $t+1$ distinct points, they are the same. Evaluating the polynomial in $X = x$, we have

$$f_S(x) = \sum_{e \in E \setminus (S \cup \{x\})} f_S(e) \prod_{u \in E \setminus (S \cup \{x, e\})} \frac{\det(x, u, S)}{\det(e, u, S)}.$$

Dividing by

$$\prod_{u \in E \setminus (S \cup \{x\})} \det(x, u, S)$$

gives

$$f_S(x) \prod_{u \in E \setminus (S \cup \{x\})} \det(x, u, S)^{-1} = - \sum_{e \in E \setminus (S \cup \{x\})} f_S(e) \prod_{u \in E \setminus (S \cup \{e\})} \det(e, u, S)^{-1},$$

from which we deduce that

$$\sum_{e \in E \setminus S} f_S(e) \prod_{u \in E \setminus (S \cup \{e\})} \det(u, S, e)^{-1} = 0,$$

and the theorem follows. □

If $k \leq p$ then the system of equations in Lemma 30 contains a set of equations which will be put to use in the classification of large arcs in this range and verifies the MDS conjecture for these values of k .

Lemma 31. *Let E be a subset of \mathcal{A} of size $k + t$ and let Δ be a subset of E of size $t + 2$. If $k \leq p$ then*

$$\sum_C g(C) \prod_{u \in E \setminus C} \det(u, C)^{-1} = 0,$$

where the sum runs over the $(k - 1)$ -subsets of E contained in Δ .

Proof. For any subset S of E of size $k - 2$, Lemma 30 implies that

$$(9) \quad \text{eqn}(S) := \sum_C g(C) \prod_{u \in E \setminus C} \det(u, C)^{-1} = 0,$$

where the sum runs over the $(k - 1)$ -subsets of E containing S . For each such S , define

$$\lambda_S = (-1)^m (k - m - 2)! m!$$

where $m = |S \cap \Delta|$, and consider the sum

$$(10) \quad \sum_S \lambda_S \text{eqn}(S),$$

where S runs over all $(k - 2)$ -subsets of E .

Let C be a $(k - 1)$ -subset of E and let $m = |C \cap \Delta|$.

If $C \not\subseteq \Delta$ then the term

$$g(C) \prod_{u \in E \setminus C} \det(u, C)^{-1}$$

appears in the sum (10) with coefficient

$$(k - 1 - m)(-1)^m (k - m - 2)! m! + m(-1)^{m-1} (k - m - 1)! (m - 1)! = 0,$$

where the first term comes from the subsets S of C for which $|S \cap \Delta| = m$ and the second term comes from the subsets S of C for which $|S \cap \Delta| = m - 1$.

If $C \subseteq \Delta$ then the term

$$g(C) \prod_{u \in E \setminus C} \det(u, C)^{-1}$$

appears with coefficient

$$(-1)^{k-2} (k - 1)!$$

from which the assertion follows. \square

9. A PROOF OF THE MDS CONJECTURE FOR PRIME FIELDS

The following theorem verifies the MDS conjecture for prime fields. We suppose throughout that $q = p^h$, for some prime p .

Theorem 32. *If $4 \leq k \leq p$ then an arc of $\text{PG}(k - 1, q)$ has at most $q + 1$ points.*

Proof. Let \mathcal{A} be an arc of $\text{PG}(k - 1, q)$ of size $q + 2 = q + k - 1 - t$, so $t = k - 3$, and define $g(C)$ as in (8). By Theorem 19, we can assume that $k \leq \frac{1}{2}q + 1$, so $k + t = 2k - 3 \leq q - 1$. Let E be a subset of \mathcal{A} of size $k + t = 2k - 3$ and let Δ be a subset of E of size $t + 2 = k - 1$. By Lemma 31,

$$g(\Delta) \prod_{u \in E \setminus \Delta} \det(u, \Delta)^{-1} = 0,$$

which is a contradiction, since $g(\Delta) \neq 0$ and $\det(u, \Delta) \neq 0$, for all $u \in \mathcal{A} \setminus \Delta$. \square

10. A PROOF OF THE MDS CONJECTURE FOR $k \leq 2p - 2$

In a similar vein to Theorem 32 we will prove that the MDS conjecture is true for $k \leq 2p - 2$.

Let \mathcal{A} be an arc of $\text{PG}(k - 1, q)$ of size $q + 2$ and define $g(C)$ as in (8). Let $X = \{x_1, \dots, x_{k-1}\}$, $Y = \{y_1, \dots, y_{k-2}\}$ be disjoint subsets of \mathcal{A} and define $E = X \cup Y$. Note that X is a set in this section and not a vector of indeterminants as in Section 8. Let r be a non-negative integer such that $r \leq k - p$ and define $X_r = \{x_1, \dots, x_r\}$ and $Y_r = \{y_1, \dots, y_r\}$. For any $T \subseteq \{1, 2, \dots, r\}$, let

$$X_T = \{x_j \mid j \in T\}, \quad Y_T = \{y_j \mid j \in T\}.$$

Define a $(k - 1)$ -subset $C_{r,T}$ of E by

$$C_{r,T} = (X \setminus X_r) \cup X_T \cup (Y_r \setminus Y_T).$$

Lemma 33. *Let \mathcal{A} be an arc of $\text{PG}(k - 1, q)$ of size $q + 2$ and suppose $k \geq p$. Let $E = X \cup Y$ be a subset of \mathcal{A} of size $2k - 3$ defined as above. Then*

$$\sum_{T \subseteq \{1, \dots, k-p\}} (-1)^{|T|} g(C_{k-p,T}) \prod_{u \in E \setminus C_{k-p,T}} \det(u, C_{k-p,T})^{-1} = 0.$$

Proof. Let S be a subset of E of size $k - 2$ and set $r = k - p$.

If $S \cap (X_r \cup Y_r) = X_T \cup (Y_r \setminus Y_T)$ for some $T \subseteq \{1, \dots, r\}$ then let

$$\lambda_S = m!(p - 2 - m)!(-1)^{m+|T|},$$

where $m = |S \cap (X \setminus X_r)|$, otherwise put $\lambda_S = 0$.

Let α_C denote the coefficient of the C -term in the sum

$$(11) \quad \sum_{S \subseteq E} \lambda_S \text{eqn}(S).$$

Then α_C is zero if $|C \cap (X_r \cup Y_r)| \notin \{r, r + 1\}$.

If $|C \cap (X_r \cup Y_r)| = r + 1$ then there exists some $i \in \{1, \dots, r\}$ for which $x_i, y_i \in C$. If i is not unique then $\lambda_S = 0$ for all $(k - 1)$ -subsets S of C , so $\alpha_C = 0$. If i is unique then

$$\alpha_C = \lambda_{C \setminus \{x_i\}} + \lambda_{C \setminus \{y_i\}} = 0.$$

If $|C \cap (X_r \cup Y_r)| = r$ and $C \cap (X_r \cap Y_r) \neq X_T \cup (Y \setminus Y_T)$, for any $T \subseteq \{1, \dots, r\}$, then $\alpha_C = 0$.

If $|C \cap (X_r \cup Y_r)| = r$ and $C \cap (X_r \cap Y_r) = X_T \cup (Y \setminus Y_T)$, for some $T \subseteq \{1, \dots, r\}$, and $C \setminus (X_r \cap Y_r) \neq X \setminus X_r$ then

$$\alpha_C = (-1)^{|T|} (m(m - 1)!(p - 1 - m)!(-1)^{m-1} + (p - 1 - m)m!(p - 2 - m)!(-1)^s) = 0$$

where $m = |C \cap (X \setminus X_r)|$.

Finally, if none of the above then $C = C_{r,T}$ for some $T \subseteq \{1, \dots, r\}$ and

$$\alpha_C = (-1)^{|T|} (k - 1 - r)(p - 2)! = (-1)^{|T|} (p - 1)!$$

which is nonzero. \square

Lemma 34. *Let C be a $(k - 1)$ -subset of \mathcal{A} . Suppose W and Δ are disjoint subsets of \mathcal{A} such that $|W| = k - |\Delta|$, and $\Delta \subseteq C$. Then*

$$\sum_{w \in W} \frac{\det(u, W \setminus w, \Delta)}{\det(w, W \setminus w, \Delta)} \det(w, C) = \det(u, C).$$

Proof. This is interpolation of the linear form $\det(u, C)$ in $u = (u_1, \dots, u_k)$ at the points of $\Delta \cup W$. \square

Theorem 35. *If q is not prime and $k \leq 2p - 2$ then an arc of $\text{PG}(k - 1, q)$ has at most $q + 1$ points.*

Proof. Let \mathcal{A} be an arc of $\text{PG}(k-1, q)$ of size $q+2$. By Theorem 32, we can assume that $k \geq p$. Let r be a non-negative integer such that $r \leq k-p$. Define $g(C)$ as in (8) and define $X, Y, E, X_r, Y_r, X_T, Y_T$ and $C_{r,T}$ as before.

We will prove that

$$\sum_{T \subseteq \{1, \dots, r\}} (-1)^{|T|} g(C_{r,T}) \prod_{u \in E \setminus C_{r,T}} \det(u, C_{r,T})^{-1} = 0$$

implies that

$$\sum_{T \subseteq \{1, \dots, r-1\}} (-1)^{|T|} g(C_{r-1,T}) \prod_{u \in E \setminus C_{r-1,T}} \det(u, C_{r-1,T})^{-1} = 0.$$

By iteration, this implies that

$$g(X) \prod_{u \in Y} \det(u, X)^{-1} = 0,$$

which is a contradiction.

By Lemma 33, the statement holds for $r = k-p$, so it suffices to show the inductive step.

Let W and Δ be disjoint subsets of $\mathcal{A} \setminus \{x_1, \dots, x_{k-1}, y_1, \dots, y_r\}$ of size $r+1$ and $k-1-r$ respectively. Let $\Omega = W \cup \{y_{2r+1}, \dots, y_{k-2}\}$. Note that $k-2 \geq 2r$, since $r \leq k-p$ and $k \leq 2p-2$, so $|\Omega| = k-1-r$. Let $w \in W$. By induction, replacing $\{y_{r+1}, \dots, y_{2r}\}$ by $W \setminus \{w\}$,

$$\sum_{T \subseteq \{1, \dots, r\}} (-1)^{|T|} g(C_{r,T}) \prod_{u \in (X \cup Y_r \cup \Omega) \setminus C_{r,T}} \det(u, C_{r,T})^{-1} \det(w, C_T) = 0.$$

Hence,

$$\sum_{T \subseteq \{1, \dots, r\}} (-1)^{|T|} g(C_{r,T}) \prod_{u \in (X \cup Y_r \cup \Omega) \setminus C_{r,T}} \det(u, C_{r,T})^{-1} \frac{\det(y_r, W \setminus w, \Delta)}{\det(w, W \setminus w, \Delta)} \det(w, C_{r,T}) = 0.$$

Summing over $w \in W$, Lemma 34 implies

$$\sum_{T \subseteq \{1, \dots, r\}} (-1)^{|T|} g(C_{r,T}) \prod_{u \in (X \cup Y_r \cup \Omega) \setminus C_{r,T}} \det(u, C_{r,T})^{-1} \det(y_r, C_{r,T}) = 0.$$

Since $\det(y_r, C_{r,T}) = 0$ if $r \notin T$, this implies that

$$\sum_{T \subseteq \{1, \dots, r-1\}} (-1)^{|T|} g(C_{r-1,T}) \prod_{u \in (X \cup Y_{r-1} \cup \Omega) \setminus C_{r-1,T}} \det(u, C_{r-1,T})^{-1} = 0.$$

This equation no longer depends on y_r , so we could have chosen y_r to be an arbitrary point of \mathcal{A} and chosen

$$\Omega = \{y_r, \dots, y_{k-2}\},$$

which proves the inductive step. \square

11. THE CLASSIFICATION OF THE LARGEST ARCS FOR $k \leq p$ AND $k \neq \frac{1}{2}(q+1)$

The proof of the following theorem exploits the system of equations in Lemma 31.

Theorem 36. *If $k \leq p$ and $k \neq \frac{1}{2}(q+1)$ then an arc of $\text{PG}(k-1, q)$ of size $q+1$ is a normal rational curve.*

Proof. Let \mathcal{A} be an arc of size $q+1$. By Corollary 19, we may assume that $k \leq \frac{1}{2}(q+1)$. Let E be a subset of \mathcal{A} of size $k+t = 2k-2$. Let Δ be a subset of E of size $t+2 = k$.

By Lemma 31,

$$\sum_C g(C) \prod_{u \in E \setminus C} \det(u, C)^{-1} = 0,$$

where the sum runs over the $(k-1)$ -subsets C of E contained in Δ .

Let W be a subset of $\mathcal{A} \setminus \Delta$ of size $k - 2$ and let $w \in W$. Let $x \in \mathcal{A} \setminus (\Delta \cup W)$ and apply the above equation with $E = \Delta \cup (W \setminus \{w\}) \cup \{x\}$. Then

$$\sum_C g(C) \prod_{u \in W} \det(u, C)^{-1} \det(\Delta \setminus C, C)^{-1} \det(x, C)^{-1} \det(w, C) = 0,$$

where the sum runs over the $(k - 1)$ -subsets of E contained in Δ .

Let D be a subset of Δ of size 3. Since $|D| = 3$ and $|W| = k - 2$, there is a linear combination v of the vectors $\{w \mid w \in W\}$ such that v is in the subspace spanned by D . Hence, $\det(v, C) = 0$ for all $(k - 1)$ -subsets C for which $D \subset C$. Thus, we can sum the above equations, so that

$$\sum_C g(C) \prod_{u \in W} \det(u, C)^{-1} \det(\Delta \setminus C, C)^{-1} \det(x, C)^{-1} \det(v, C) = 0,$$

where the sum is over the three $(k - 1)$ -subsets of Δ which do not contain D .

With respect to the basis $\Delta = \{e_1, \dots, e_k\}$, ordered so that the elements of D are the first three elements of the basis, the above sum is

$$\frac{a_1}{x_1} + \frac{a_2}{x_2} + \frac{a_3}{x_3} = 0,$$

where

$$a_i = g(\Delta \setminus \{e_i\}) \prod_{u \in W} \det(u, \Delta \setminus \{e_i\})^{-1} \det(e_i, \Delta \setminus \{e_i\})^{-1} \det(v, \Delta \setminus \{e_i\}).$$

Thus we have that for all $x \in \mathcal{A} \setminus W$,

$$a_1 x_2 x_3 + a_2 x_1 x_3 + a_3 x_1 x_2 = 0.$$

The coefficients a_1 , a_2 and a_3 are determined, up to scalar factor, by two points of $\mathcal{A} \setminus (W \cup \Delta)$. Thus, since

$$|\mathcal{A}| - (2k - 2) \geq 3,$$

we can switch an element of W with an element of $\mathcal{A} \setminus (W \cup \Delta)$ and conclude that

$$a_1 x_2 x_3 + a_2 x_1 x_3 + a_3 x_1 x_2 = 0,$$

for all $x \in \mathcal{A}$.

This implies that the projection of \mathcal{A} from $\Delta \setminus D$ is contained in a conic, for any subsets Δ and D of \mathcal{A} . By Corollary 25, \mathcal{A} is a normal rational curve. \square

12. EXTENDING SMALL ARCS TO LARGE ARCS

Let \mathcal{G} be an arc of $\text{PG}(k - 1, q)$ arbitrarily ordered. Suppose that \mathcal{G} can be extended to an arc \mathcal{A} of $\text{PG}(k - 1, q)$ of size $q + k - 1 - t \geq k + t$. Let $n = |\mathcal{G}| - k - t$ be a non-negative integer.

Let P_n be the matrix whose columns are indexed by the $(k - 1)$ -subsets C of \mathcal{G} and whose rows are indexed by pairs (S, U) , where S is a $(k - 2)$ -subset of \mathcal{G} and U is a n -subset of $\mathcal{G} \setminus S$. The $((S, U), C)$ entry of P_n is zero unless C contains S in which case it is $\prod_{u \in U} \det(u, C)$.

The following theorem is from [19], see also [3].

Theorem 37. *If an arc \mathcal{G} of $\text{PG}(k - 1, q)$ can be extended to an arc of size $q + 2k - 1 - |\mathcal{G}| + n$ then the system of equations $P_n v = 0$ has a solution in which all the coordinates of v are non-zero.*

Proof. Let $|\mathcal{G}| = k + t + n$ and suppose that \mathcal{G} extends to an arc \mathcal{A} of size $q + k - 1 - t$.

Let U be a subset of \mathcal{G} of size n . Then $E = \mathcal{G} \setminus U$ is a subset of \mathcal{G} of size $k + t$. By Lemma 30, for each subset S of E of size $k - 2$,

$$\sum_C g(C) \prod_{u \in \mathcal{G} \setminus (U \cup C)} \det(u, C)^{-1} = 0,$$

where the sum runs over all $(k-2)$ -subsets C of E containing S .

Then

$$\sum_C \prod_{u \in U} \det(u, C) \left(g(C) \prod_{u \in \mathcal{G} \setminus C} \det(u, C)^{-1} \right) = 0.$$

This system of equations is given by the matrix P_n and a solution v is a vector with C coordinate

$$\alpha_{C, \mathcal{G}} = g(C) \prod_{u \in \mathcal{G} \setminus C} \det(u, C)^{-1}$$

which is non-zero for all $(k-1)$ -subsets C of \mathcal{G} . \square

Suppose that we do find a solution v to the system of equations. Then we know the value of $\alpha_{C, \mathcal{G}}$ and therefore $f_S(x)$, where $C = S \cup \{x\}$. This would allow one to calculate the polynomials $f_S(X)$ for each subset S of \mathcal{G} of size $k-2$. Therefore, if \mathcal{G} does extend to an arc \mathcal{A} then each solution tells us precisely the tangent hyperplanes to \mathcal{A} containing S , for each $(k-2)$ -subset S of \mathcal{G} .

By starting with a sub-arc \mathcal{G} of size $3k-6$ of the normal rational curve one can prove that the matrix P_n has full rank and conclude the following theorem, see [4].

Theorem 38. *If \mathcal{G} is a subset of the normal rational curve of $\text{PG}(k-1, q)$ of size $3k-6$ and q is odd, then \mathcal{G} cannot be extended to an arc of size $q+2$.*

13. THE SEGRE-BLOKHUIS-BRUEN-THAS HYPERSURFACE

The *Segre-Blokhuis-Bruen-Thas hypersurface* associated to an arc \mathcal{A} in $\text{PG}(k-1, q)$ is a hypersurface in the dual of $\text{PG}(k-1, q)$ containing the set of points which are dual to the hyperplanes of $\text{PG}(k-1, q)$ meeting \mathcal{A} in $k-2$ points. Its existence was proved in a sequence of papers by Segre ([57]) and Blokhuis, Bruen and Thas ([9] and [10]). The following theorem gives an alternative proof. It relies on the scaled coordinate-free lemma of tangents (Lemma 29).

For $i \in \{1, \dots, k-1\}$ denote by X_i the $(k-1)$ -tuple $(X_{i1}, X_{i2}, \dots, X_{ik-1})$ and by $\det_j(X_1, \dots, X_{k-1})$ the determinant in which the j -coordinate of each of the X_i 's has been deleted.

Theorem 39. *Let $m \in \{1, 2\}$ such that $m-1 = q$ modulo 2. If \mathcal{A} is an arc of $\text{PG}(k-1, q)$ of size $q+k-1-t$, where $|\mathcal{A}| \geq mt+k-1$, then there is a homogeneous polynomial in k variables $\phi(Z)$, of degree mt , which gives a polynomial $G(X_1, \dots, X_{k-1})$ under the substitution $Z_j = \det_j(X_1, \dots, X_{k-1})$, with the property that for all $\{y_1, \dots, y_{k-2}\} \subset \mathcal{A}$*

$$G(X, y_1, \dots, y_{k-2}) = f_{\{y_1, \dots, y_{k-2}\}}(X)^m.$$

Moreover, $\phi(Z)$ is unique.

Proof. Order the arc \mathcal{A} arbitrarily and let E be a subset of \mathcal{A} of size $mt+k-1$. Define

$$(12) \quad G(X_1, \dots, X_{k-1}) = \sum_T (f_{T \setminus \{a_{k-1}\}}(a_{k-1}))^m \prod_{u \in E \setminus T} \frac{\det(X_1, \dots, X_{k-1}, u)}{\det(a_1, \dots, a_{k-1}, u)},$$

where the sum runs over subsets $T = \{a_1, \dots, a_{k-1}\}$ of E .

Observe that G can be obtained from a homogeneous polynomial of degree mt in Z_1, \dots, Z_k under the change of variables $Z_j = \det_j(X_1, \dots, X_{k-1})$.

For $S = \{y_1, \dots, y_{k-2}\}$, define

$$h_S(X) := G(y_1, \dots, y_{k-2}, X).$$

Note that $h_S(X)$ is well-defined since any reordering of S can only ever multiply $h_S(X)$ by $(-1)^{mt} = 1$.

For $S \subset E$, the only nonzero terms in $h_S(X)$ are obtained for subsets T of E containing S . Therefore,

$$h_S(X) = \sum_{a \in E \setminus S} (f_S(a))^m \prod_{u \in E \setminus (S \cup \{a\})} \frac{\det(y_1, \dots, y_{k-2}, X, u)}{\det(y_1, \dots, y_{k-2}, a, u)}.$$

The evaluation of $h_S(X)$ at $x \in E$ is equal to zero if $x \in S$ and equal to $(f_S(x))^m \neq 0$ otherwise. Since, with respect to a basis containing S both f_S^m and h_S are homogeneous polynomials in two variables of degree mt and $E \setminus S$ has size $mt + 1$, we may conclude that $h_S = f_S^m$.

If S is not contained in E then we proceed by induction on the size of $S \setminus E$. As induction hypothesis we assume that for each subset S with $S \setminus E$ of size r the polynomials h_S and f_S^m are equal. Let $S = \{y_1, \dots, y_{k-2}\}$ be such that $S \setminus E$ is of size $r + 1$. W.l.o.g. assume $y_{k-1} \notin E$. Then for $x \in E$ we have

$$h_S(x) = (-1)^{mt} h_{S'}(y_{k-1}) = h_{S'}(y_{k-1}),$$

where S' is the set obtained from S by replacing the $(k-1)$ -th element y_{k-1} of S by x . On the other hand, by the definition (8) of g and Lemma 29, the scaled coordinate-free lemma of tangents, we have

$$(f_S(x))^m = (g(y_1, \dots, y_{k-1}, x))^m = (g(y_1, \dots, y_{k-2}, x, y_{k-1}))^m = (f_{S'}(y_{k-1}))^m.$$

By induction, $h_{S'}(y_{k-1}) = (f_{S'}(y_{k-1}))^m$, and therefore the polynomials h_S and f_S^m have the same evaluation at points in E . Applying the same argument as in the case where $S \subset E$ we obtain $h_S = f_S^m$.

Denote by $\phi(Z)$ the polynomial obtained from $G(X_1, \dots, X_{k-1})$ under the substitution

$$Z_j = \det_j(X_1, \dots, X_{k-1}),$$

$j = 1, \dots, k$. Then $\phi(Z)$ has degree mt . Consider any $S = \{y_1, \dots, y_{k-2}\} \subset \mathcal{A}$ and any point x contained in the tangent hypersurface $\mathcal{Z}(f_S)$ of \mathcal{A} at S . Then $G(x, y_1, \dots, y_{k-2}) = f_S(x)^m = 0$. This implies that $\phi(z) = 0$, where $z = (z_1, \dots, z_k)$ and

$$z_j = \det_j(x, y_1, \dots, y_{k-2}).$$

Therefore, the polynomial $\phi(Z)$ vanishes at the set of points, denoted by \mathcal{T} , of the dual space which are dual to the hyperplanes meeting \mathcal{A} in $k-2$ points. It remains to prove that ϕ is unique. Suppose that $\phi(Z)$ and $\phi'(Z)$ are two forms of degree mt vanishing on \mathcal{T} .

We will prove by induction on r that for a subset D of \mathcal{A} of size $k-r$, there is a constant $a \in \mathbb{F}_q$, such that the restriction of $\phi - a\phi'$ to $\langle D \rangle^\perp$ is zero. Once this is established uniqueness follows when $r = k$.

For $r = 2$, $\langle D \rangle^\perp$ is a line containing t points of \mathcal{T} . Both ϕ and ϕ' have zeros of multiplicity m at these t points. Thus, selecting a so that $\phi - a\phi'$ is zero at a point distinct from these t points, the polynomial $\phi - a\phi'$ is zero at $mt+1$ points (counting with multiplicity) of the line $\langle D \rangle^\perp$, which implies it is zero on all the points of $\langle D \rangle^\perp$.

Now, suppose that D is a set of $k-r$ points. By induction, for $x \in \mathcal{A} \setminus D$, we can choose an $a_x \in \mathbb{F}_q$ so that $\phi - a_x\phi'$ is zero on $\langle D \cup \{x\} \rangle^\perp$. For two distinct points $x, y \in \mathcal{A} \setminus D$ the form $a_x\phi' - a_y\phi'$ therefore vanishes on the two hyperplanes $\langle D \cup \{x\} \rangle^\perp$ and $\langle D \cup \{y\} \rangle^\perp$ of $\langle D \rangle^\perp$. Since ϕ' does not vanish on their intersection (there exists a hyperplane through D , x and y which is spanned by \mathcal{A}) it follows that $a_x = a_y$. So there exists an $a \in \mathbb{F}_q$ for which the form $\phi - a\phi'$ vanishes on every hyperplane of $\langle D \rangle^\perp$ of the form $\langle D \cup \{x\} \rangle^\perp$, with $x \in \mathcal{A} \setminus D$. By hypothesis there are at least $mt+1$ such hyperplanes. It follows that $\phi - a\phi'$ must be zero on $\langle D \rangle^\perp$. This proves the inductive step. \square

The following theorem is a corollary of the uniqueness part of Theorem 39. In the planar case this is due to Segre [57] and the general case is due to Blokhuis, Bruen and Thas [9] and [10].

Theorem 40. *Let $m \in \{1, 2\}$ such that $m-1 = q$ modulo 2. If \mathcal{A} is an arc of $\text{PG}(k-1, q)$ of size at least $mq/(m-1) + k-1$ then \mathcal{A} has a unique completion to a complete arc.*

Proof. Let $\phi(Z)$ be the form of degree mt obtained from \mathcal{A} by Theorem 39. Suppose that \mathcal{A} can be extended to a larger arc by appending the point x . In the dual space the hyperplane x^\perp contains the point dual to the hyperplane $\langle S \cup \{x\} \rangle^\perp$ for every $(k-2)$ -subset S of \mathcal{A} . By induction on r , we will prove that ϕ is zero on $\langle D \cup x \rangle^\perp$ for every subset D of \mathcal{A} of size $k-2-r$. Then, for $r = k-2$, this will imply that ϕ is zero on x^\perp . Thus, ϕ contains linear factors of multiplicity m for each point x which extends \mathcal{A} to a larger arc. Thus, the extension of \mathcal{A} to a complete arc is unique and can be found by finding the linear factors of $\phi(Z)$.

To prove the induction claim for $r = 1$, let D be a $(k-3)$ -subset of \mathcal{A} . Then, for every point $y \in \mathcal{A} \setminus D$, the point $\langle D \cup \{x, y\} \rangle^\perp$ is a point on the line $\langle D \cup \{x\} \rangle^\perp$, which is a zero of ϕ of multiplicity m . Since ϕ has degree mt , this implies that ϕ is zero on the line $\langle D \cup \{x\} \rangle^\perp$.

To prove the inductive step, consider a $(k-2-r)$ -subset D of \mathcal{A} and observe that for every point $y \in \mathcal{A} \setminus D$, the form ϕ is zero on $\langle D \cup \{x, y\} \rangle^\perp$. Thus, ϕ is zero on more than mt hyperplanes of $\langle D \cup \{x\} \rangle^\perp$, which implies ϕ is zero on $\langle D \cup \{x\} \rangle^\perp$. \square

The following theorem is a corollary to Theorem 39.

Theorem 41. *Let $m \in \{1, 2\}$ such that $m-1 = q$ modulo 2. Let \mathcal{A} be an arc of $\text{PG}(k-1, q)$ of size $q+k-1-t$, where $|\mathcal{A}| \geq mt+k-1$. Let \mathcal{T} be the set of points in the dual space which are dual to the hyperplanes of $\text{PG}(k-1, q)$ which contain exactly $k-2$ points of \mathcal{A} . Then \mathcal{T} is contained in a unique hypersurface of degree mt .*

Proof. The hypersurface is defined by $\phi(Z) = 0$, where $\phi(Z)$ is as in Theorem 39. \square

14. A TENSOR ASSOCIATED TO AN ARC

The following theorem is in the spirit of Theorem 39. It is stronger in that one does not require $m = 2$ in the case of q odd, but weaker in the sense that the multi-homogeneous form F does not necessarily come from a form $\phi(Z)$ on $\text{PG}(k-1, q)$ under substitution. This theorem is from [5] and it is proved using the scaled coordinate-free lemma of tangents (Lemma 29).

Theorem 42. *Let \mathcal{A} be an arc of $\text{PG}(k-1, q)$ of size $q+k-1-t$ and let $\Psi[X]$ denote the subspace of homogenous polynomials of degree t which are zero on \mathcal{A} . There is a form $F(X_1, \dots, X_{k-1})$, which is homogenous of degree t in each of the indeterminates $X_i = (X_{i1}, \dots, X_{ik})$ with the property that for each $S = \{y_1, \dots, y_{k-2}\} \subset \mathcal{A}$*

$$F(X, y_1, \dots, y_{k-2}) = f_S(X) \pmod{\Psi[X]}.$$

Moreover, modulo $(\Psi[X_1], \dots, \Psi[X_{k-1}])$, the form F is alternating for t even and symmetric for t odd.

One can show that for large subsets of \mathcal{A} such a form exists such that the statement holds without the modulo $\Psi[X]$. We conjecture that the statement indeed holds without the modulo $\Psi[X]$.

The multi-homogeneous form F from Theorem 42 is called a *tensor form* of \mathcal{A} . Here is an example.

Example 43. The set of 12 points in $\text{PG}(2, 13)$,

$$\begin{aligned} \mathcal{A} = \{ & (3, 4, 1), (-3, 4, 1), (3, -4, 1), (-3, -4, 1), (4, 3, 1), (4, -3, 1), (-4, 3, 1), (-4, -3, 1), \\ & (1, 1, 1), (1, -1, 1), (-1, 1, 1), (-1, -1, 1) \} \end{aligned}$$

is an arc with $t = 3$ and it is not contained in a curve of degree 3. Consequently, Theorem 42 implies that there is bi-homogeneous form $F(X, Y)$ of degree three with the property that $F(X, a) = f_a(X)$ for all $a \in S$. It is given by

$$\begin{aligned} F(x, y) = & 5(x_2^2 x_3 y_1^2 y_3 + y_2^2 y_3 x_1^2 x_3 + x_2 x_3^2 y_1^2 y_2 + x_1^2 x_2 y_2 y_3^2 + x_1 x_3^2 y_1 y_2^2 + x_1 x_2^2 y_1 y_3^2) \\ & + 6x_1 x_2 x_3 y_1 y_2 y_3 + x_1^3 y_1^3 + x_2^3 y_2^3 + x_3^3 y_3^3. \end{aligned}$$

In this example the tensor form F of \mathcal{A} is unique up to a scalar factor.

We will see that Theorem 42 has strong implications for planar arcs and leads to a proof of the MDS conjecture for $k \leq \sqrt{q} - \sqrt{q}/p + 2$ in the case that q is an odd square.

The tensor form for planar arcs is a bi-homogeneous form of bi-degree (t, t) , which we call a (t, t) -form. To be able to state the next theorem we need the following terminology. Let $V_r[X]$ denote the vector space of homogeneous polynomials of degree r in $\mathbb{F}_q[X_1, X_2, X_3]$. For a set D of points of $\text{PG}(2, q)$, we define an r -socle of D as follows. Let $M_r(D)$ denote the matrix whose rows are indexed by the elements of $V_r[X]$ and whose columns are indexed by the points of D , and where the $(f(X), x)$ -entry is $f(x)$. Then an r -socle of D is a subset of D whose elements index the columns which form a basis for the column space of $M_r(D)$.

The following theorem from [5] applies to planar arcs and is more detailed than Theorem 42 for the tensor form of an arc in higher dimensions.

Theorem 44. *Let \mathcal{A} be an arc of size $q + 2 - t$ of $\text{PG}(2, q)$ and let $\Psi[X]$ denote the subspace of homogeneous polynomials of degree t which are zero on \mathcal{A} . For any subset S of \mathcal{A} containing a t -socle S_0 of \mathcal{A} , where*

$$|S \setminus S_0| = \min\{\frac{1}{2}(t+2)(t+1), |\mathcal{A} \setminus S_0|\},$$

there is a (t, t) -form $F(X, Y) \in \mathbb{F}_q[X, Y]$ such that

$$F(X, y) = f_y(X) \pmod{\Psi[X]}$$

for all $y \in \mathcal{A}$ and

$$F(X, y) = f_y(X),$$

for all $y \in S$. Moreover, modulo $(\Psi[X], \Psi[Y])$, the form F is alternating for t even and symmetric for t odd.

As we will see in the section on planar arcs, Theorem 44 leads to the existence of certain algebraic curves containing a given planar arc.

15. PLANAR ARCS

It follows from the trivial upper bound Theorem 13 that a planar arc can have size at most $q + 2$, and by the existence of a hyperoval, this is best possible for q even. However, when q is odd this can easily be improved.

Theorem 45. *If q is odd then a planar arc has size at most $q + 1$.*

Proof. As we have seen in the proof of Theorem 13, considering a point $x \in S$ and the $q + 1$ lines through x , the bound $|\mathcal{A}| \leq q + 2$ follows immediately. Now suppose $|\mathcal{A}| = q + 2$. Then each line through a point in \mathcal{A} must contain another point in \mathcal{A} , and therefore each line intersects \mathcal{A} in 0 or 2 points. Counting points on the lines through a point $y \notin S$ implies that $|\mathcal{A}|$, and hence q , is even. \square

The following theorem is Segre's celebrated characterisation of conics from [52]. We give a proof using the tensor associated to a planar arc.

Theorem 46. *If q is odd and \mathcal{A} is an arc of size $q + 1$ in $\text{PG}(2, q)$ then \mathcal{A} is a conic.*

Proof. Applying Theorem 44, where in this case $t = 1$ and $\Psi[X] = \{0\}$, we obtain a symmetric bilinear form $F(X, Y)$ with $F(a, a) = 0$ for each $a \in \mathcal{A}$. \square

15.1. **Hyperovals.** The situation is very different for q even. There are many examples of arcs of size $q + 2$ known for q even which are projectively inequivalent. These can be described by a so-called *o-polynomial* which is a polynomial f with the property that

$$\{(1, t, f(t)) \mid t \in \mathbb{F}_q\} \cup \{(0, 1, 0), (0, 0, 1)\}$$

is an arc of size $q + 2$. A planar arc of size $q + 2$ is called a *hyperoval*.

The list of hyperovals is complete for $q = 8$, $q = 16$ and $q = 64$. There is a sporadic example when $q = 32$ due to O’Keefe and Penttila [43].

Table 1 lists all the known families of hyperovals. The references for these infinite classes of hyperovals are: (1) regular (Bose [11]), (2) translation (Segre [54]), (3) Segre [56], (4) Glynn I and Glynn II [24], (4) Payne [45], (5) Cherowitzo [16], (6) Subiaco (Cherowitzo, Penttila, Pinneri and Royle [18], Payne [46], Payne, Penttila and Pinneri [47]), (7) Adelaide (Cherowitzo, O’Keefe and Penttila [17]).

Name	$f(X)$	$q = 2^h$	Conditions
Regular	X^2		
Translation	X^{2^i}		$(h, i) = 1$
Segre	X^6	h odd	
Glynn I	$X^{3\sigma+4}$	h odd	$\sigma = 2^{(h+1)/2}$
Glynn II	$X^{\sigma+\lambda}$	h odd	$\sigma = 2^{(h+1)/2};$ $\lambda = 2^m$ if $h = 4m - 1;$ $\lambda = 2^{3m+1}$ if $h = 4m + 1$
Payne	$P(X)$	h odd	
Cherowitzo	$C(X)$	h odd	$\sigma = 2^{(h+1)/2}$
Subiaco I	$S_1(X)$	$h = 4r + 2$	$\omega^2 + \omega + 1 = 0$
Subiaco II	$S_2(X)$	$h = 4r + 2$	$\delta = \zeta^{q-1} + \zeta^{1-q},$ ζ primitive in \mathbb{F}_{q^2}
Subiaco III	$S_3(X)$	$h \neq 4r + 2$	$T_2(1/\delta) = 1$
Adelaide	$S(X)$	h even, $h \geq 4$	$\beta \in \mathbb{F}_{q^2} \setminus \{1\}, \beta^{q+1} = 1,$ $m \equiv \pm(q-1)/3 \pmod{q+1}$

Table 1: Hyperovals in $\text{PG}(2, q)$, q even

In Table 1, T_2 denotes the trace function from \mathbb{F}_q to \mathbb{F}_2 and

$$P(X) = X^{1/6} + X^{3/6} + X^{5/6},$$

$$C(X) = X^\sigma + X^{\sigma+2} + X^{3\sigma+4}$$

$$S_1(X) = \frac{\omega^2(X^4 + X)}{X^4 + \omega^2 X^2 + 1} + X^{1/2},$$

$$S_2(X) = \frac{\delta^2 X^4 + \delta^5 X^3 + \delta^2 X^2 + \delta^3 X}{X^4 + \delta^2 X^2 + 1} + \left(\frac{X}{\delta}\right)^{1/2},$$

$$S_3(X) = \frac{(\delta^4 + \delta^2)X^3 + \delta^3 X^3 + \delta^2 X}{X^4 + \delta^2 X^2 + 1} + \left(\frac{X}{\delta}\right)^{1/2},$$

and

$$S(X) = \frac{T(\beta^m)(X+1)}{T(\beta)} + \frac{T((\beta X + \beta^q)^m)}{T(\beta)(X + T(\beta)X^{1/2} + 1)^{m-1}} + X^{1/2},$$

where $T(X) = X + X^q$.

The complete classification of hyperovals has been determined for $q \leq 64$ and is given in the next theorem.

- Theorem 47.** (i) (Segre [54]) *In $\text{PG}(2, q)$, $q = 2, 4, 8$, the only hyperovals are the regular hyperovals.*
- (ii) (Hall [28], O’Keefe and Penttila [42]) *In $\text{PG}(2, 16)$, there are exactly two distinct hyperovals. They are the regular hyperoval and the Subiaco hyperoval. This latter hyperoval is also called the Lunelli-Sce hyperoval [41].*
- (iii) (Penttila and Royle [49]) *In $\text{PG}(2, 32)$, there are exactly six distinct hyperovals. They are the regular hyperoval, the translation hyperoval, the Segre hyperoval, the Payne hyperoval, the Cherowitzo hyperoval and the O’Keefe–Penttila hyperoval.*
- (iv) (Vandendriessche [60]) *In $\text{PG}(2, 64)$, there are exactly four distinct hyperovals. They are the regular hyperoval, the Subiaco I, the Subiaco II and the Adelaide hyperoval.*

The following theorem is due to Caullery and Schmidt and is from [15].

Theorem 48. *If f is an o -polynomial of \mathbb{F}_q of degree less than $\frac{1}{2}q^{1/4}$ then $f(X)$ is equivalent to either X^6 or X^{2^k} for a positive integer k .*

This also classifies *exceptional o -polynomials* extending results from [30] and [64] on monomial o -polynomials.

15.2. Second largest planar arcs, q even. The following theorem implies that the second largest complete arc is somewhat smaller than the size of a hyperoval. Its proof illustrates the power of the algebraic hypersurface (in this case a curve) of an arc from the planar version of Theorem 39.

Theorem 49. *If q is even then a planar arc of size at least $q - \sqrt{q} + \frac{3}{2}$ in $\text{PG}(2, q)$ is extendable to a hyperoval.*

Proof. Let $t = q + 2 - |\mathcal{A}| \leq \sqrt{q} + \frac{1}{2}$, and observe that \mathcal{A} satisfies the hypothesis of Theorem 40 (since q is even, $m = 1$). It follows that \mathcal{A} is uniquely extendable to a complete arc.

So we may assume that \mathcal{A} is a complete arc of size at least $q - \sqrt{q} + \frac{3}{2}$. Suppose \mathcal{A} has size less than $q + 2$. Then $t \geq 1$. Let $\phi(Z)$ be the form of degree t obtained from \mathcal{A} by Theorem 39. If $\phi(Z)$ has a linear factor then by the arguments used in the proof of Theorem 40, \mathcal{A} can be extended to a larger arc, a contradiction. Hence $\phi(Z)$ does not have linear factors. Let T denote the set of points on the curve defined by $\phi(Z)$. Then $T \neq \emptyset$, and each line meets T in at most t points. Choose $x \in T$. Counting points of T on the lines through x we obtain $|T| \leq (t - 1)(q + 1) + 1$. Since the tangents to \mathcal{A} form a set of $|\mathcal{A}|t = (q + 2 - t)t$ points in the dual plane, we have

$$(q + 2 - t)t \leq (t - 1)(q + 1) + 1,$$

which contradicts $t \leq \sqrt{q} + \frac{1}{2}$. □

In [57] Segre used the algebraic curve in the dual plane of a planar arc (which he called the *algebraic envelope*) in combination with the Hasse-Weil bound to obtain the upper bound $q - \sqrt{q} + 1$ for a planar arc in $\text{PG}(2, q)$, q even. This bound is sharp as we will see later.

Theorem 50. *If q is even then a complete planar arc in $\text{PG}(2, q)$ which is not contained in a hyperoval has size at most $q - \sqrt{q} + 1$.*

The following theorem is from Voloch [62]. Its proof combines Theorem 39 with the Stöhr-Voloch theorem.

Theorem 51. *If q is an even non-square then a planar arc of size larger than $q - \sqrt{2q} + 2$ is extendable to a hyperoval.*

15.3. Second largest planar arcs, q odd. The bounds on the size of the second largest arc in planes of odd order, obtained by the methods described above, are worse than for q even. The reason for this is the fact that the hypersurface from Theorem 39 has degree $2t$ for q odd. As with the proof of Theorem 51, the theorems by Voloch use Theorem 41 and the Stöhr-Voloch theorem from [58]. The following theorems are from [61] and [62] respectively.

Theorem 52. *If q is prime then a planar arc of size larger than $\frac{44}{45}q + \frac{8}{9}$ is contained in a conic.*

Theorem 53. *If q is an odd non-square then a planar arc of size larger than $q - \frac{1}{4}\sqrt{pq} + \frac{29}{16}p - 1$ is contained in a conic.*

In contrast to the previous bounds the following theorem does not rely on Hasse-Weil or Stöhr-Voloch. It is proved using the tensor of a planar arc Theorem 44 from [5]. It illustrates that in this case the tensor form (of degree t in each component) is stronger than the algebraic hypersurface (which is of degree $2t$).

Theorem 54. *Let \mathcal{A} be a planar arc of size $q + 2 - t$ not contained in a conic. If q is odd then \mathcal{A} is contained in the intersection of two curves, sharing no common component, each of degree at most $t + p^{\lfloor \log_p t \rfloor}$.*

Theorem 54 has the following corollary.

Theorem 55. *If q is an odd square then a planar arc of size at least $q - \sqrt{q} + \sqrt{q}/p + 3$ is contained in a conic.*

In the case that q is a non-square and non-prime, the bound obtained from Theorem 54 does not improve upon the bound of Voloch mentioned above. However, in the case that q is prime, it does improve on Voloch's bound for primes less than 1783, see [5, Theorem 5].

15.4. Planar arcs of size $q - \sqrt{q} + 1$. In the case that q is a square, Kestenband [40] constructed large complete planar arcs as the intersection of Hermitian curves. For any 3×3 matrix M , let

$$V(M) = \{x \in \text{PG}(2, q) \mid x^t M x^{\sqrt{q}} = 0\}.$$

Theorem 56. *Let $q > 4$ be a square. Let I be the 3×3 identity matrix. Let $H = (a_{ij})$ be a 3×3 matrix with the property that $H^{\sqrt{q}} = H^t$, where $H^{\sqrt{q}} = (a_{ij}^{\sqrt{q}})$ and $H^t = (a_{ji})$.*

If the characteristic polynomial of H is irreducible over \mathbb{F}_q then the intersection of the Hermitian curves $\mathcal{A} = V(I) \cap V(H)$ is a planar arc of size $q - \sqrt{q} + 1$ which is not contained in a conic.

Proof. Consider the Hermitian curves $V(H + \mu I)$, where $\mu \in \mathbb{F}_{\sqrt{q}}$ and $V(I)$. If x is a point on two of these curves then $x \in \mathcal{A}$. If $x \notin \mathcal{A}$ and $x \notin V(I)$ then x is a point of $V(H + (a/b)I)$, where $x^t H x^{\sqrt{q}} = a$ and $x^t I x^{\sqrt{q}} = -b$. Hence, each point is either in \mathcal{A} or on exactly one of the $\sqrt{q} + 1$ Hermitian curves. Therefore,

$$(\sqrt{q} + 1)(q\sqrt{q} + 1) = |\mathcal{A}|(\sqrt{q} + 1) + q^2 + q + 1 - |\mathcal{A}|,$$

which gives $|\mathcal{A}| = q - \sqrt{q} + 1$.

Suppose that ℓ is a line incident with $r \geq 2$ points of \mathcal{A} . Then ℓ intersects each Hermitian curve ($V(H + \mu I)$ or $V(I)$) in $\sqrt{q} + 1$ points, r of which are in \mathcal{A} and $\sqrt{q} + 1 - r$ of which are not in \mathcal{A} .

Counting points of ℓ not in \mathcal{A} we have $(\sqrt{q} + 1 - r)(\sqrt{q} + 1) = q + 1 - r$, since each point not in \mathcal{A} is on exactly one of the $\sqrt{q} + 1$ Hermitian curves. This gives $r = 2$ and so \mathcal{A} is an arc.

It follows from Bezout's theorem that \mathcal{A} has at most $2\sqrt{q} + 2$ points in common with a conic, so cannot be contained in a conic for $q - \sqrt{q} + 1 \geq 2\sqrt{q} + 2$. The case $q = 9$ can be checked directly. \square

15.5. **Classification of planar arcs of size $\geq q-2$.** Planar arcs of size q were shown to be incomplete by Segre for q odd and by Tallini for q even. In contrast there do exist complete planar arcs of size $q-1$. The classification of planar arcs of size $q-1$ and $q-2$ was only recently completed in [5] and is given in the following corollaries.

Corollary 57 proves a conjecture from Hirschfeld, Korchmáros and Torres [34, Remark 13.63]. It was proven in [5] as a corollary of Theorem 54.

Corollary 57. *The only complete planar arcs of size $q-1$ occur for $q=7$ (there are 2 projectively distinct arcs of size 6), for $q=9$ (there is a unique arc of size 8), $q=11$ (there is a unique arc of size 10) and $q=13$ (there is a unique arc of size 12).*

Corollary 58. *The only complete planar arcs of size $q-2$ occur for $q=8$ (there are 3 projectively distinct arcs of size 6), for $q=9$ (there is a unique arc of size 7) and $q=11$ (there are 3 projectively distinct arcs of size 9).*

16. A PROOF OF THE MDS CONJECTURE FOR $k \leq \sqrt{q} - \sqrt{q}/p + 2$

Suppose that q is an odd square. In the previous section we saw how the tensor for planar arcs has far reaching consequences for the size of planar arcs which are not contained in conics. Using the projection theorem this leads to a proof of the MDS conjecture for $k \leq \sqrt{q} - \sqrt{q}/p + 2$. The following two corollaries are from [5].

Corollary 59. *If $k \leq \sqrt{q} - \sqrt{q}/p + 1$ and $q = p^{2h}$, p odd, then an arc of $\text{PG}(k-1, q)$ of size $q+1$ is a normal rational curve.*

Proof. Let \mathcal{A} be an arc of size $q+1$ in $\text{PG}(k-1, q)$. The projection of \mathcal{A} from a subset of size $k-3$ is a planar arc of size at least $q+1 - (\sqrt{q} - \sqrt{q}/p - 2)$. By Theorem 55, this projection is contained in a conic. By Corollary 25, this implies that \mathcal{A} is a normal rational curve. \square

Corollary 60. *If $k \leq \sqrt{q} - \sqrt{q}/p + 2$ and $q = p^{2h}$, p odd, then an arc of $\text{PG}(k-1, q)$ has size at most $q+1$.*

Proof. Suppose \mathcal{A} is an arc of size $q+2$ in $\text{PG}(k-1, q)$. By Corollary 59, the projection of \mathcal{A} from any of its points is a normal rational curve. By Theorem 24, the arc \mathcal{A} is contained in a normal rational curve, a contradiction. \square

17. ARCS IN $\text{PG}(3, q)$

The following theorem was proved by Segre [53] for q odd and by Casse [12] for q even. The theorem proves the MDS conjecture for $k=4$.

Theorem 61. *An arc of $\text{PG}(3, q)$, $q \geq 4$, has size at most $q+1$.*

Proof. If q is odd and \mathcal{A} is an arc in $\text{PG}(3, q)$ of size $q+2$, then by projection from a point of \mathcal{A} we obtain an arc of size $q+1$ in a plane, which by Segre's theorem is a conic. By projecting \mathcal{A} from two distinct points and applying Theorem 24 it follows that \mathcal{A} is contained in a unique normal rational curve of $\text{PG}(3, q)$. This contradicts the assumption $|\mathcal{A}| = q+2$.

Let \mathcal{A} be an arc of size $q+2$ with q even. By Theorem 41, the set of points dual to the planes containing exactly two points of \mathcal{A} is contained in a hypersurface of degree one (a plane). Hence, these planes are all concurrent in a point y , which extends \mathcal{A} to an arc of size $q+3$.

Let \mathcal{A}' be an arc of size $q+3$. The number of points which are incident with bisecants to \mathcal{A}' is

$$q+3 + \binom{q+3}{2}(q-1).$$

This is less than $(q+1)(q^2+1)$, the total number of points of $\text{PG}(3, q)$, which implies there is a point u not incident with any bisecant. Since $|\mathcal{A}'| = q+3$, any plane incident with two points of \mathcal{A}' is incident with three points of \mathcal{A}' . Therefore, the number of planes incident with u and three points of \mathcal{A}' is

$$\frac{1}{3} \binom{q+3}{2},$$

which implies 3 divides $q+2$.

Let x and y be points of \mathcal{A}' spanning the line ℓ and let v be a point of $\ell \setminus \{x, y\}$. For each pair of points $z, w \in \mathcal{A}' \setminus \{x, y\}$, the plane spanned by v, z and w contains three points of $\mathcal{A}' \setminus \{x, y\}$. Thus, the number of 3-secant planes which are incident with v but do not contain x or y is

$$\frac{1}{3} \binom{q+1}{2}.$$

This implies 3 divides $q+1$, contradicting the previously obtained divisibility condition. \square

The following theorem classifies arcs of size $q+1$ in $\text{PG}(3, q)$. The case q odd was already obtained by Segre [53]. The case q even is due to Casse and Glynn [13].

Theorem 62. *Let \mathcal{A} be an arc in $\text{PG}(3, q)$ of size $q+1$. If q is odd and $q \geq 5$ then \mathcal{A} is a twisted cubic. If q is even and $q \geq 8$ then \mathcal{A} is projectively equivalent to Example 10.*

Proof. Suppose q is odd. Projecting \mathcal{A} from two distinct points gives to planar arcs of size q , each of which, by Theorems 55, 52, 53, is contained in a conic. Applying Theorem 24 concludes the proof.

Suppose $q = 2^h$. Projecting \mathcal{A} from a point $x \in \mathcal{A}$ gives a planar arc $x(\mathcal{A})$ of size q . By Theorem 50, $x(\mathcal{A})$ is extendable to a hyperoval. Let a, b be the unique pair of points which extend $x(\mathcal{A})$ to this hyperoval. The lines xa and xb are called the *tangent lines of \mathcal{A} at the point x* . For any point $y \neq x$ of \mathcal{A} , it follows from Theorem 41, that the planes xya and xyb are contained in a hyperbolic quadric in the dual space. Therefore the lines xa and xb are contained in a hyperbolic quadric in the dual space. Hence, the $2(q+1)$ tangent lines of \mathcal{A} are the lines on a hyperbolic quadric Q^+ in $\text{PG}(3, q)$. We may assume that Q^+ is the zero locus of the quadratic form $X_1X_4 - X_2X_3$ and that $x = (1, 0, 0, 0)$ and $y = (0, 0, 0, 1)$ are in \mathcal{A} . The ovals $x(\mathcal{A})$ and $y(\mathcal{A})$ correspond to two *o-polynomials* f and g with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. The arc \mathcal{A} can therefore be written as

$$\mathcal{A} = \{(rg(r), g(r), r, 1) : r \in \mathbb{F}_q\} \cup \{x\} = \{(1, s, f(s), sf(s)) : s \in \mathbb{F}_q\} \cup \{y\}.$$

Equating coefficients in the *o-polynomials* f and g , leads to the condition that $f(1/s) = g(s)$, for each $s \neq 0$, and consequently to the conclusion that the only possibility for f and g is $f(s) = g(s) = s^n$ for some integer n . So

$$\mathcal{A} = \{(1, s, s^n, s^{n+1}) : s \in \mathbb{F}_q\} \cup \{(0, 0, 0, 1)\}.$$

It follows that \mathcal{A} is invariant under the projectivity φ interchanging the first with the last and the second with the third coordinate. In particular φ interchanges the points x and y . Since x and y were arbitrary points of \mathcal{A} it follows that \mathcal{A} is invariant under a group $G \leq \text{PGL}(4, q)$, which acts transitively on the points of \mathcal{A} . Since the tangent plane at x (which has equation $X_4 = 0$) intersects \mathcal{A} in x with multiplicity $n+1$, each tangent plane must intersect \mathcal{A} in a point with multiplicity $n+1$. Imposing this condition on the tangent plane at a general point $(1, s, s^n, s^{n+1})$ implies that $n = 2^e$ for some integer e . The fact that \mathcal{A} is an arc implies $(e, h) = 1$ (see last line of the proof of Example 10). \square

18. ARCS IN $\text{PG}(4, q)$

The following theorem proves the MDS conjecture for $k = 5$. It was proved by Segre [57] for q odd and Casse [12] for q even.

Theorem 63. *An arc \mathcal{A} in $\text{PG}(4, q)$, $q \geq 5$, has size at most $q + 1$.*

Proof. Let \mathcal{A} be an arc in $\text{PG}(4, q)$ of size $q + 2$.

Suppose q is odd. Projecting \mathcal{A} from any two distinct points of \mathcal{A} gives a planar arc of size q , which is contained in a conic. By Corollary 25, the arc \mathcal{A} is contained in a normal rational curve, a contradiction.

Suppose q is even. Projecting from two distinct points $x, y \in \mathcal{A}$ gives two arcs $x(\mathcal{A})$ and $y(\mathcal{A})$ of size $q + 1$ in a hyperplane π of $\text{PG}(4, q)$. Arguing as in the proof of Theorem 62, it follows that for each of these projected arcs the $2(q + 1)$ tangent lines are the lines on a hyperbolic quadric Q_x^+ and Q_y^+ in π . In particular, $x(\mathcal{A})$ is contained in Q_x^+ and $y(\mathcal{A})$ is contained in Q_y^+ .

Let z be the point of intersection of the line through x and y with π and let ℓ be one of the tangent lines of $x(\mathcal{A})$ at z . By the definition of ℓ , each plane through ℓ contains at most one point of $x(\mathcal{A})$ different from z . For each $p \in \mathcal{A} \setminus \{x, y\}$ the plane $\langle x, y, p \rangle$ meets π in a line containing z , $x(p)$ and $y(p)$. Therefore, the plane spanned by ℓ and $x(p)$ also contains $y(p)$. Hence, each plane through ℓ also contains at most one point of $y(\mathcal{A})$ different from z . It follows that the two tangents, say ℓ and m , of $x(\mathcal{A})$ at z are equal to the two tangents of $y(\mathcal{A})$ at z .

The quadratic cones xQ_x^+ and yQ_y^+ therefore coincide on the hyperplane $\langle x, \ell, m \rangle$ with the union of the two planes $\langle x, \ell \rangle$ and $\langle x, m \rangle$. After a suitable choice of basis, we can assume that xQ_x^+ and yQ_y^+ are the zero locus of the forms $X_1X_2 + X_5a(X)$ and $X_1X_2 + X_5b(X)$, where $a(X)$ and $b(X)$ are linear forms in $X = (X_1, \dots, X_5)$. The arc \mathcal{A} is therefore contained in the zero locus of $X_5(a(X) - b(X))$, which is the union of two hyperplanes. This implies that \mathcal{A} has size at most 8, a contradiction. \square

The following improves on the previous bound $q > 83$ (see e.g. [35, Table 3.1]).

Theorem 64. *An arc in $\text{PG}(4, q)$, q odd, $q > 13$, of size $q + 1$ is a normal rational curve.*

Proof. Projecting from two points gives a planar arc of size $q - 1$ which by Theorem 57 is contained in a conic unless $q \leq 13$. \square

Remark 65. *An arc \mathcal{A} of size 8 in $\text{PG}(4, 7)$ gives a 5-dimensional MDS code $C_{\mathcal{A}}$ of length 8, whose dual is a 3-dimensional MDS code of length 8, which by Segre's theorem is a Reed-Solomon code. By Theorem 20, the code $C_{\mathcal{A}}$ is also a Reed-Solomon code, and therefore \mathcal{A} is a normal rational curve. In [23] Glynn proved that there are exactly two arcs of size 10 in $\text{PG}(4, 9)$.*

To prove the following theorem from [14], Casse and Glynn prove that every arc of size q in $\text{PG}(3, q)$, q even and $q \geq 16$, is incomplete and its completion is unique. From this one can then prove that an arc of size $q + 1$ in $\text{PG}(4, q)$, q even and $q \geq 16$, is a normal rational curve.

Theorem 66. *An arc in $\text{PG}(4, q)$, $q = 2^h$, $h \geq 3$, of size $q + 1$ is a normal rational curve.*

19. STATE OF THE ART ON THE MDS CONJECTURE

Proofs of the MDS conjecture for k -dimensional codes with $k \in \{4, 5\}$ can be found in Sections 17 and 18. The following table contains a survey of the conditions on the parameters for $k \geq 6$ for which the MDS conjecture has been proven.

Reference	condition on q	condition on k
Casse - Glynn 1984	$q = 2^h$	$k = 6$
Kaneta - Maruta 1991	$q = 2^h$	$k = 7$
Storme - Thas 1993	$q = 2^h$	$k < \frac{1}{2}\sqrt{q} + \frac{15}{4}$
Voloch 1991	$q = p^{2h+1}$ odd	$k < \frac{1}{4}\sqrt{pq} - \frac{29}{16}p + 4$
Ball - Lavrauw 2018	$q = p^{2h}$ odd	$k < \sqrt{q} - \sqrt{q}/p + 2$
Hirschfeld - Korchmáros 1996	$q = p^h, p \geq 5$ odd	$k \leq \frac{1}{2}\sqrt{q}$
Hirschfeld - Korchmáros 1998	$q \geq 529$ odd, $q \neq 3^6, 5^5$	$k \leq \frac{1}{2}\sqrt{q} + 2$
Ball 2012	$q = p^h$	$k < p$
Ball - De Beule 2012	$q = p^h, h > 1$	$k \leq 2p - 2$

Parameters for which the MDS conjecture holds true in $\text{PG}(k-1, q)$ with $k \geq 6$

The proof of the MDS conjecture for $k = 6$, q even, follows from Theorem 66 and Corollary 26.

In 1991 Kaneta and Maruta [39] proved the uniqueness of arcs of size $q+1$ in $\text{PG}(5, q)$, q even, $q \geq 16$, which by Corollary 26 proves the MDS conjecture for $k = 7$ and these values of q .

In [59] the Segre-Blokhuis-Bruen-Thas hypersurface is used to obtain an upper bound for the second largest arc in $\text{PG}(4, q)$, q even, improving upon the bound from [9]. By Theorem 66, any arc exceeding that bound is contained in a normal rational curve. Using Corollaries 25 and 26, this proves the MDS conjecture for $k < \frac{1}{2}\sqrt{q} + \frac{15}{4}$.

The following instances for the MDS conjecture for q odd follow from the results on planar arcs and the projection theorem. Precisely, projecting an arc \mathcal{A} in $\text{PG}(k-2, q)$ of size $q+1$ from $k-4$ of its points gives a planar arc of size $q-k+5$. So if $q+5-k > m$, i.e. $k < q+5-m$, and each planar arc of size $> m$ is contained in a conic then, by Corollary 25, \mathcal{A} is a normal rational curve. By Corollary 26 the MDS conjecture holds true in $\text{PG}(k-1, q)$.

The best asymptotic bounds for q odd are from [62] and [5]. Theorem 53 implies the MDS conjecture for $k < \frac{1}{4}\sqrt{pq} - \frac{29}{16}p + 4$, q an odd non-square. If q is an odd square then Theorem 55 implies the MDS conjecture for $k < \sqrt{q} - \sqrt{q}/p + 2$.

For a proof of the MDS conjecture for $k \leq p$ and $k \leq 2p-2$, we refer to Sections 9 and 10.

The bounds on planar arcs from Hirschfeld and Korchmáros [32] and [33] imply the MDS conjecture for $k \leq \frac{1}{2}\sqrt{q}$, provided that the characteristic is at least 5, and for $k \leq \frac{1}{2}\sqrt{q} + 2$, for $q \geq 529$ and $q \neq 3^6, 5^5$.

We have not dealt here very deeply with the techniques used to obtain the results of Voloch [62] and Hirschfeld and Korchmáros [32], [33]. These results not only use bounds on the number of points of algebraic curves of small degree but also require more careful analysis of the curves which can arise from Segre's algebraic envelope associated to the tangents of a planar arc.

REFERENCES

- [1] S. Ball, On sets of vectors of a finite vector space in which every subset of basis size is a basis, *J. Eur. Math. Soc.*, **14** (2012) 733–748.
- [2] S. Ball, *Finite Geometry and Combinatorial Applications*, London Mathematical Society Student Texts **82**, Cambridge University Press, 2015.
- [3] S. Ball, Extending small arcs to large arcs, *European Journal of Mathematics*, **4** (2018) 8–25.
- [4] S. Ball and J. De Beule, On subsets of the normal rational curve, *IEEE Trans. Information Theory*, **63** (2017) 3658–3662.
- [5] S. Ball and M. Lavrauw, Planar arcs, *J. Combin. Theory Ser. A*, **160** (2018) 261–287.
- [6] S. Ball and M. Lavrauw, Arcs and tensors, *Des. Codes and Cryptogr.*, to appear.
- [7] A. Barg and I. Tamo, A Family of Optimal Locally Recoverable Codes, *IEEE Trans. Inform. Theory*, **60** 4661–4676 (2014).
- [8] A. Bernal, On the Existence of Absolutely Maximally Entangled States of Minimal Support, *Quant. Phys. Lett.*, **6** 1-3 (2017).
- [9] A. Blokhuis, A. A. Bruen and J. A. Thas, On M.D.S. codes, arcs in $PG(n, q)$ with q even, and a solution of three fundamental problems of B. Segre, *Invent. Math.* **92** (1988) 441–459.
- [10] A. Blokhuis, A. A. Bruen and J. A. Thas, Arcs in $PG(n, q)$, MDS-codes and three fundamental problems of B. Segre - some extensions, *Geom. Dedicata*, **35** (1990) 1–11.
- [11] R.C. Bose, Mathematical theory of the symmetrical factorial design, *Sankhyā* **8** (1947), 107–166.
- [12] L. R. A. Casse, A solution to B. Segre’s problem $I_{r,q}$, *Atti Accad. Naz. Lincei Rend.*, **46** (1969) 13–20.
- [13] L. R. A. Casse and D. G. Glynn, The solution to Beniamino Segre’s problem $I_{r,q}$, $r = 3$, $q = 2^h$, *Geom. Dedicata*, **13** (1982) 157–163.
- [14] L. R. A. Casse and D. G. Glynn, On the uniqueness of $(q + 1)$ -arcs of $PG(4, q)$, $q = 2^h$, $h \geq 3$, *Discrete Math.*, **48** (1984) 173–186.
- [15] F. Caullery and K-U Schmidt, On the classification of hyperovals, *Adv. Math.*, **283** (2015) 195–203.
- [16] W.E. Cherowitzo, α -flocks and hyperovals, *Geom. Dedicata* **72** (1998), 221–246.
- [17] W.E. Cherowitzo, C.M. O’Keefe and T. Penttila, A unified construction of finite geometries in characteristic two, *Advances in Geometry* **3** (2003) 1-21.
- [18] W.E. Cherowitzo, T. Penttila, I. Pinneri and G.F. Royle, Flocks and ovals, *Geom. Dedicata* **60** (1996), 17–37.
- [19] A. Chowdhury, Inclusion Matrices and the MDS Conjecture, *Electronic J. Combin.*, **23** (2016) P4.29.
- [20] E. Crestani and A. Lucchini, d -Wise generation of prosolvable groups, *J. Algebra*, **369** (2012) 59–69.
- [21] O. Farràs, C. Padró, C. Xing and A. Yang, Natural generalizations of threshold secret sharing, *IEEE Trans. Inform. Theory*, **60** (2014) 1652–1664.
- [22] M. Giulietti, F. Pambianco, F. Torres, and E. Ughi, On large complete arcs: odd case, *Discrete Math.*, **225** (2002) 145–159.
- [23] D. G. Glynn, The non-classical 10-arc of $PG(4, 9)$, *Discrete Math.*, **59** (1986) 43–51.
- [24] D.G. Glynn, Two new sequences of ovals in finite Desarguesian planes of even order, *Combinatorial Mathematics X* (ed. L.R.A. Casse), Lecture Notes in Mathematics **1036**, Springer, 1983, 217–229.
- [25] M. Grassl and M. Rötteler, Quantum MDS codes over small fields, in *Proc. Int. Symp. Inf. Theory (ISIT)*, 1104–1108 (2015).
- [26] A. Gray, T. J. Reid, X. Zhou, Clones in matroids representable over a prime field, *Discrete Math.* **341** (2018) 213–216.
- [27] V. Guruswami, C. Xing, C. Yuan, How long can optimal locally repairable codes be?, *IEEE Trans. Inform. Theory*, **65** (2019) 3662–3670.
- [28] M. Hall, Ovals in the Desarguesian plane of order 16, *Ann. Mat. Pura Appl.* **102** (1975), 159–176.
- [29] J. Harris. Algebraic Geometry, A First Course.
- [30] F. Hernando and G. M. McGuire, Proof of a conjecture of Segre and Bartocci on monomial hyperovals in projective planes, *Des. Codes Cryptogr.* **65** (2012), no. 3, 275–289
- [31] J. W. P. Hirschfeld, Rational curves on quadrics over finite fields of characteristic two, *Rend. Mat.*, **3** (1971) 772-795.
- [32] J. W. P. Hirschfeld and G. Korchmáros, On the embedding of an arc into a conic in a finite plane, *Finite Fields Appl.*, **2** (1996) 274–292.
- [33] J. W. P. Hirschfeld and G. Korchmáros, On the number of rational points on an algebraic curve over a finite field, *Bull. Belg. Math. Soc. Simon Stevin*, **5** (1998) 313–340.
- [34] J. W. P. Hirschfeld, G. Korchmáros and F. Torres, Algebraic curves over a finite field. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2008. xx+696 pp. ISBN: 978-0-691-09679-7
- [35] J. W. P. Hirschfeld and L. Storme. The packing problem in statistics, coding theory and finite projective spaces: update 2001. Finite geometries, 201–246, Dev. Math., 3, Kluwer Acad. Publ., Dordrecht, 2001.
- [36] S. Huntemann, The upper bound of general Maximum Distance Separable codes, Honours thesis, University of New Brunswick, St. John, Canada, 2012.
- [37] , G. Järnefelt and P. Kustaanheimo, An observation on finite geometries. Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 166?182. Johan Grundt Tanums Forlag, Oslo, 1952.

- [38] H. Kaneta and T. Maruta, An elementary proof and an extension of Thas' theorem on k -arcs, *Math. Proc. Camb. Phil. Soc.* **105** (1989) 459–462.
- [39] T. Maruta and H. Kaneta, On the uniqueness of $(q + 1)$ -arcs of $PG(5, q)$, $q = h$, $h \geq 4$, *Math. Proc. Cambridge Philos. Soc.* 110 (1991), no. 1, 91–94.
- [40] B. C. Kestenband, Unital intersections in finite projective planes, *Geom. Dedicata*, **11** (1981) 107–117.
- [41] L. Lunelli and M. Sce, Considerazione aritmetiche e risultati sperimentali sui $\{K; n\}_q$ -archi, *Ist. Lombardo Accad. Sci. Rend. A* **98** (1964), 3–52.
- [42] C.M. O'Keefe and T. Penttila, Hyperovals in $PG(2, 16)$, *European J. Combin.* **12** (1991), 51–59.
- [43] C.M. O'Keefe and T. Penttila, A new hyperoval in $PG(2, 32)$, *J. Geom.* **44** (1992), 117–139.
- [44] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland, 1977.
- [45] S.E. Payne, A new infinite family of generalized quadrangles, *Congr. Numer.* **49** (1985), 115–128.
- [46] S.E. Payne, A tensor product action on q -clan generalized quadrangles with $q = 2^e$, *Linear Algebra Appl.* **226–228** (1995), 115–137.
- [47] S.E. Payne, T. Penttila and I. Pinneri, Isomorphisms between Subiaco q -clan geometries, *Bull. Belg. Math. Soc.* **2** (1995), 197–222.
- [48] T. Penttila and I. Pinneri, Hyperovals, *Australasian J. Combin.*, **19** (1999) 101–114.
- [49] T. Penttila and G.F. Royle, Classification of hyperovals in $PG(2, 32)$, *J. Geom.* **50** (1994), 151–158.
- [50] I. Pinneri, Flocks, Generalised Quadrangles and Hyperovals, PhD dissertation, University of Western Australia, (1996).
- [51] V. N. Potapov, On the number of SQSs, Latin hypercubes and MDS codes, *J. Combin. Des.*, **26** (2018) 237–248.
- [52] B. Segre, Ovals in a finite projective plane, *Canad. J. Math.*, **7** (1955) 414–416.
- [53] B. Segre, Curve razionali normali e k -archi negli spazi finiti, *Ann. Mat. Pura Appl.*, **39** (1955) 357–379.
- [54] B. Segre, Sui k -archi nei piani finiti di caratteristica due, *Rev. Math. Pures Appl.* **2** (1957), 289–300.
- [55] B. Segre, Le geometrie di Galois. *Ann. Mat. Pura Appl.* (4) **48** (1959), 1–96.
- [56] B. Segre, Ovali e curve σ nei piani di Galois di caratteristica due, *Atti dell' Accad. Naz. Lincei Rend.* (8) **32** (1962), 785–790.
- [57] B. Segre, Introduction to Galois geometries, *Atti Accad. Naz. Lincei Mem.*, **8** (1967) 133–236.
- [58] K. O. Stöhr and J. F. Voloch, Weierstrass points on curves over finite fields, *Proc. London Math. Soc.*, **52** (1986) 1–19.
- [59] L. Storme and J. A. Thas, M.D.S. codes and arcs in $PG(n, q)$ with q even: an improvement of the bounds of Bruen, Thas, and Blokhuis. (English summary) *J. Combin. Theory Ser. A* 62 (1993), no. 1, 139–154.
- [60] P. Vandendriessche, Classification of the hyperovals in $PG(2, 64)$, *Electronic J. Combin.*, **26** Issue 2 (2019).
- [61] J. F. Voloch, Arcs in projective planes over prime fields, *J. Geom.*, **38** (1990) 198–200.
- [62] J. F. Voloch, Complete arcs in Galois planes of non-square order, in: *Advances in Finite Geometries and Designs*, Oxford University Press, Oxford, 1991, pp. 401–406.
- [63] X. Wang, L. Ji, Y. Li, Yun and M. Liang, Constructions of augmented orthogonal arrays, *J. Combin. Des.*, **26** (2018) 547–559.
- [64] M. E. Zieve, Planar functions and perfect nonlinear monomials over finite fields, *Des. Codes Cryptogr.* **75** (2015), no. 1, 71–80.