The number of directions determined by a function over a finite field

Simeon Ball *
Department of Mathematics,
Queen Mary College, University of London,
London, E1 4NS,
United Kingdom

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Abstract

A proof is presented that shows that the number of directions determined by a function over a finite field $GF(q)$ is either 1, at least $(q+3)/2$, or between $q/s+1$ and $(q-1)/(s-1)$ for some s where $GF(s)$ is a subfield of $GF(q)$. Moreover the graph of those functions that determine less than half the directions is $GF(s)-$linear. This completes the unresolved cases $s=2$ and $s=3$ of the main theorem in [1].

1. Introduction

Let $GF(q)$ be the finite field with $q=p^n$ elements, p prime. Let f be a function from $GF(q)$ to $GF(q)$, let

$$D := \left\{ \frac{f(y)-f(x)}{y-x} \;\middle|\; x,y \in GF(q),\ x \neq y \right\}$$

be the set of directions determined by the function f and $N=|D|$. Rédei and Megyesi [7] proved that either $N=1$, $N \geq (q+3)/2$ or $(q+s)/(s+1) \leq N \leq (q-1)/(s-1)$ for some $s=p^e \leq \sqrt{q}$. In [2] their proof was simplified and in [1] sharp bounds on N were proven in almost all cases. This article contains a proof, which also covers the unresolved cases, and concludes that either $N=1$, $N \geq (q+3)/2$ or $q/s+1 \leq N \leq (q-1)/(s-1)$ for some s where $GF(s)$ is a subfield of $GF(q)$.

A few remarks are in order before proceeding. Firstly, any set of q points of $AG(2,q)$ that do not determine all the directions are affinely equivalent to the graph of a function. If c is an element of $GF(q)$ not determined as a direction by the function f then $f(X)-cX$ is a permutation polynomial [3]. The set of points

$$B = \{(x,f(x)) \mid x \in GF(q)\} \cup \{(c) \mid c \in D\},$$

where (c) is the point at infinity incident with the lines with direction c, is a blocking set of $PG(2,q)$ of size $q+N$. Moreover, any blocking set of $PG(2,q)$ of size $q+N \leq 2q$ with an N-secant can be constructed from the graph of a function.

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The objective of this article is to prove the following theorem. This theorem is similar to [1, Theorem 1.1]. The difference being that in the case when $s = 2$, the lower bound on N has been improved from $(q+5)/3$ to $q/2+1$ and in the case when $s = 3$, the graph of f is shown to be $GF(3)$-linear. The proof presented here will use many of the basic ideas from [1], [2] and [7] along with Hasse derivatives, which have not previously been used to attack this problem.

Theorem 1.1 *Let f be a function from $GF(q)$ to $GF(q)$, $q = p^h$ for some prime p, and let N be the number of directions determined by f. Let $s = p^e$ be maximal such that any line with a direction determined by f that is incident with a point of the graph of f is incident with a multiple of s points of the graph of f. One of the following holds:*

(i) $s = 1$ *and* $(q+3)/2 \leq N \leq q+1$;

(ii) $GF(s)$ *is a subfield of* $GF(q)$ *and* $q/s+1 \leq N \leq (q-1)/(s-1)$;

(iii) $s = q$ *and* $N = 1$.

Moreover if $s > 2$ then the graph of f is $GF(s)$-linear.

Before proving the lower bounds, note first that the upper bounds are trivial. Let P be a point of the graph of f. If $s > 1$ then the points on the graph of f lie on lines incident with P and with a direction in D. Therefore

$$N(s-1)+1 \leq q.$$

We will require the following lemma on lacunary polynomials which comes from [7, Satz 18] (also [1, Lemma 2]). The degree of a polynomial g is denoted by g°.

Lemma 1.2 *Let s be a power of p with $1 \leq s < q$ and suppose that*

$$X^{q/s} + g(X) \in GF(q)[X] \setminus GF(q)[X^p]$$

is fully reducible over $GF(q)$. Either $s = 1$ and $g(X) = -X$ or

$$g^\circ \geq (q+s)/s(s+1).$$

Proof : A zero of $X^{q/s}+g$ of multiplicity k is a zero of $(X^{q/s}+g)' = g'$ of multiplicity at least $k-1$ and a zero of $X^q - X$ with multiplicity 1. Hence

$$X^{q/s}+g \;\Big|\; ((X^{q/s}+g)^s - (X^q - X))g' = (g^s + X)g' \ .$$

By assumption $g' \neq 0$ so either $g^s = -X$ and hence $s = 1$ or the right hand side is non-zero and

$$q/s \leq sg^\circ + g^\circ - 1.$$

□

The following lemma will also be required.

Lemma 1.3 *Let s be a power of p with $1 \le s < q$ and suppose that*

$$f = X^{q/s} + g \in GF(q)[X] \setminus GF(q)[X^p]$$

is fully reducible over $GF(q)$. If $s > 2$, $g^\circ = q/s^2$ and $2(g')^\circ < g^\circ$ then

$$f \in \langle 1, X, X^s, X^{s^2}, \dots, X^{q/s} \rangle_{(GF(q))}.$$

Proof : We have the same divisibility as in the previous Lemma. Let $m(X)$ be the quotient of this division. Now $m^\circ = (g')^\circ$ where

$$(X^{q/s} + g)m = (X + g^s)g'.$$

We differentiate with respect to X to obtain another equation and combine the two equations to deduce that

$$X^{q/s}(g''m - m'g') + (gm)g'' - (gm)'g' = -(g')^2.$$

The degree of $(gm)g'' - (gm)'g'$ is less than $3g^\circ < q/s$ since $s \ge 3$ so $g''m = m'g'$ and $(gm)'g' - (gm)g'' = (g')^2$. Eliminating g'' from these equations gives $(m-1)g' = 0$ and hence $m = 1$. It follows that the degree of g' is zero and from the differential equation it is an easy task to deduce that f is in the required span. □

The essential difference between the (partial) proof of Theorem 1.1 presented in [1], which focussed strongly on lacunary polynomials, and the proof presented here, is the use of Hasse derivatives.

The *k-th Hasse derivative with respect to Y* is defined on a polynomial

$$f(Y) = \sum_{i=0}^{n} b_i Y^i \quad \text{by} \quad H_k(f(Y)) := \sum_{i=0}^{n} \binom{i}{k} b_i Y^{i-k}.$$

Hasse derivatives have the following property:

$$H_k\left(\prod_{i=1}^{n}(Y + a_i)\right) = \sum_{(a_1,a_2,\dots,a_k)} \left(\prod_{i=k+1}^{n}(Y + a_i)\right),$$

where the sum is taken over all k-tuples.

The proof of Theorem 1.1 will take the next four sections, however, only the first two sections are required to prove the bounds on N.

2. The Rédei polynomial

Throughout f is a function from $GF(q)$ to $GF(q)$, D is the set of directions determined by f, $N = |D|$ is the number of these directions, and $s = p^e$ is maximal such that any line with a direction in D, that is incident with a point of the graph of f, is incident with a multiple of s points.

The Rédei polynomial is a polynomial in two variables defined as

$$R(X,Y) := \prod_{x \in GF(q)} (X + xY - f(x)) = \sum_{j=0}^{q} \sigma_j(Y) X^{q-j}.$$

The polynomial in one variable $R(X,y)$ has a repeated factor if and only if there exists $x \neq z \in GF(q)$ such that $xy-f(x) = zy-f(z)$ if and only if $y = (f(x)-f(z))/(x-z) \in D$. Hence for $y \notin D$

$$R(X,y) = X^q - X,$$

and $\sigma_j(y) = 0$ for $j = q$ and $j = 1,2,\ldots,q-2$.

The overall degree of $R(X,Y)$ is q and so the degree of $\sigma_j(Y)$ is at most j. Moreover the coefficient of Y^j in $\sigma_j(Y)$ is the coefficient of X^{q-j} in the homogeneous part of degree q in $R(X,Y)$, which is

$$\prod_{x \in GF(q)} (X+xY) = X^q - XY^{q-1}.$$

Hence for $j = 1,2,\ldots,q-2$ the degree of $\sigma_j(Y)$ is at most $j-1$ and moreover, from the previous paragraph, has at least $q-N$ distinct zeros. The polynomial $\sigma_j(Y) \equiv 0$ whenever $j \leq q-N$. Therefore

$$R(X,Y) = X^q + \sum_{j=0}^{N-1} \sigma_{q-j}(Y)X^j.$$

Let $y \in D$. The line $c = yX - Y$ is incident with exactly k points of the graph of f if and only if there exist exactly k distinct $x \in GF(q)$ such that $c = xy - f(x)$ which is if and only if $X + c$ is a factor of $R(X,y)$ exactly k times. By assumption, every line with direction y that is incident with a point of the graph of f is incident with a multiple of s points of the graph, so every factor of $R(X,y)$ is a factor a multiple of s times. i.e. $R(X,y)$ is an s-th power. Extracting s-th roots we can write $R(X,y)^{1/s} = X^{q/s} + g(X)$. Since s is chosen to be maximal there exists a $y \in D$ such that $g'(X) \neq 0$. It follows from Lemma 1.2 that, for any $y \in D$ such that $g'(X) \neq 0$, $g^\circ \geq (q+s)/s(s+1)$. Let N_0 be maximal such that $\sigma_{q-N_0} \not\equiv 0$. We have shown that

$$\frac{q+s}{(s+1)} \leq N_0 \leq N-1.$$

The completes the proof of Rédei and Megyesi for the case $s = 1$. In the remainder of the article $s > 1$.

If $j \neq 1$ and s does not divide j then $\sigma_j(y) = 0$ for all $y \in GF(q)$ and so it follows that $\sigma_j(Y) \equiv 0$. Therefore

$$R(X,Y) = X^q + \sum_{j=0}^{N_0/s} \sigma_{q-js}(Y)X^{js} + \sigma_{q-1}(Y)X.$$

3. The Hasse derivatives of $R(X,Y)$

Let us take the k-th Hasse derivative of $R(X,Y)$ with respect to Y.

$$H_k(R(X,Y)) = \sum_{(x_1,x_2,\ldots,x_k)} x_1x_2\ldots x_k \prod_{x\neq x_i}(X+xY-f(x))$$

This can be rewritten as

$$H_k(R(X,Y)) = \sum_{(x_1,x_2,\ldots,x_k)} \left(\frac{x_1}{X+x_1Y-f(x_1)}\right)\cdots\left(\frac{x_k}{X+x_kY-f(x_k)}\right)R(X,Y). \quad (1)$$

For all $x \in GF(q)$ and $y \in D$

$$X + xy - f(x) \;\Big|\; R(X,y) - (X^q - X) = X + \sum_{j=0}^{N_0/s} \sigma_{q-js}(y) X^{js} \,.$$

Note that $\sigma_{q-1}(y) = 0$ for $y \in D$ since $R(X,y)$ is an s-th power.

If we evaluate the equation (1) for $y \in D$ and multiply by the polynomial above k times then we get an equality of polynomials in X the right-hand side of which contains a factor $R(X,y)$ which divides the left-hand side. This is the divisibility

$$R(X,y) \;\Bigg|\; \left(X + \sum_{j=0}^{N_0/s} \sigma_{q-js}(y) X^{js} \right)^k H_k(R)(X,y) \,,$$

where $H_k(R)(X,y)$ denotes the k-th Hasse derivative of R with respect to Y evaluated at $Y = y$. The polynomial

$$H_k(R)(X,y) = \sum_{j=0}^{N_0/s} H_k(\sigma_{q-js})(y) X^{js} + H_k(\sigma_{q-1})(y) X$$

has degree at most N_0 so the right-hand side of the divisibility has degree at most $(k+1)N_0$.

The polynomial

$$X + \sum_{j=0}^{N_0/s} \sigma_{q-js}(y) X^{js}$$

is not zero as this would imply that $s = 1$. If $(k+1)N_0 \leq q-1$ then the left-hand side of the divisibility has degree larger than the right-hand side and we conclude that

$$H_k(\sigma_{q-js})(y) = 0$$

for all $0 \leq j \leq N_0/s$.

Let us assume that $N \leq q/s$.

Now $N_0 \leq N-1 \leq q/s - 1$ and $(k+1)N_0 \leq sN_0 \leq q-s < q-1$ whenever $k \leq s-1$. Hence for $k \leq s-1$ the k-th Hasse derivatives of the polynomials σ_{q-js} are zero when evaluated at $y \in D$. We shall show that they are in fact identically zero when $js = N_0$.

It follows immediately from the definition of Hasse derivatives that $H_{s-1}(f)$ is an s-th power for any polynomial f. A zero of an s-th power is a zero of multiplicity at least s. Hence there are at least sN zeros of the polynomial $H_{s-1}(\sigma_{q-N_0})$. However the degree of this polynomial is less than $q - N_0 < sN_0 < sN$. Hence $H_{s-1}(\sigma_{q-N_0}) \equiv 0$. But $H_{s-2}(\sigma_{q-N_0})$ is then an s-th power, it also has at least sN zeros and is therefore identically zero. We continue by induction. Assume that $H_{s-i}(\sigma_{q-N_0}) \equiv 0$ for $i = 1, 2, \ldots, j-1$. Then $H_{s-j}(\sigma_{q-N_0})$ is an s-th power and if $j < s$ it has sN zeros and is therefore identically zero. Finally for $j = s$ we have that σ_{q-N_0} itself is an s-th power. However σ_{q-N_0} is zero for all $y \notin D$ and so has at least $s(q-N)$ zeros which is far greater than its degree. But $\sigma_{q-N_0} \not\equiv 0$ by assumption, a contradiction.

Hence $N \geq q/s + 1$.

4. $GF(s)$ is a subfield of $GF(q)$

Without loss of generality we can assume that $f(0) = 0$.

Note that

$$H_k(R)(X, y)' = H_k(\sigma_{q-1})(y),$$

where f' is the derivative of f with respect to X, and note that the polynomial

$$\sigma_{q-1}(Y) = \prod_{x \in GF(q)^*} (xY - f(x)).$$

Moreover y is a zero of σ_{q-1} exactly k times if and only if the line joining (y) to $(0, 0)$ contains exactly k points, other than $(0, 0)$, of the graph of f.

Let $D^\infty \subset D$ be such that for all $y \in D^\infty$ the point (y) is joined to $(0, 0)$ by an s-secant. Then $H_k(\sigma_{q-1})(y) = 0$ for $k = 0, 1, \ldots, s - 2$ and $H_{s-1}(\sigma_{q-1})(y) \neq 0$. Hence $H_{s-1}(R)(X, y) \neq 0$.

Counting points of the graph of f on lines incident with the origin

$$|D \setminus D^\infty| \leq \frac{q - 1 - N(s - 1)}{s} \leq \frac{q}{s^2} - 1.$$

Hence $|D^\infty| \geq N - q/s^2 + 1$.

Let $Q(X)$, a polynomial of degree at most $sN_0 - q$, be the quotient in the divisibility

$$R(X, y) \;\Bigg|\; \left(X + \sum_{j=0}^{N_0/s} \sigma_{q-js}(y) X^{js} \right)^{s-1} H_{s-1}(R)(X, y)\ ,$$

where $y \in D^\infty$ and so $H_{s-1}(R)(X, y) \neq 0$. Differentiating with respect to X we get

$$R(X, y)Q'(X) = \left(X + \sum_{j=0}^{N_0/s} \sigma_{q-js}(y) X^{js} \right)^{s-1} H_{s-1}(\sigma_{q-1})(y)$$

$$- \left(X + \sum_{j=0}^{N_0/s} \sigma_{q-js}(y) X^{js} \right)^{s-2} H_{s-1}(R)(X, y).$$

The right-hand side of this equation has degree at most $(s - 1)N_0 < q$ while if $Q' \neq 0$ then the left hand side has degree at least q. We conclude $Q' = 0$ and

$$H_{s-1}(R)(X, y) = H_{s-1}(\sigma_{q-1})(y) \left(X + \sum_{j=0}^{N_0/s} \sigma_{q-js}(y) X^{js} \right).$$

Extracting s-th roots and incorporating any constant multiple in the quotient Q we have

$$R^{1/s} Q^{1/s} = X + \sum_{j=0}^{N_0/s} \sigma_{q-js}(y) X^{js}.$$

Differentiating again with respect to X we see $R^{1/s}(Q^{1/s})' + (R^{1/s})'Q^{1/s} = 1$. Now $((R^{1/s})'Q^{1/s})^\circ \leq N_0/s - 1 + N_0 - q/s < q/s$ and the degree of $R^{1/s}$ is q/s so $(Q^{1/s})' = 0$. Therefore $Q^{1/s}$ and $(R^{1/s})'$ are constant and hence for some $\gamma \in GF(q)$

$$R(X, y) = \gamma \left(X + \sum_{j=0}^{N_0/s} \sigma_{q-js}(y) X^{js} \right)^s.$$

Now it is a simple matter to check that since

$$R(X, y) = X^q + \sum_{j=0}^{N_0/s} \sigma_{q-js}(y) X^{js}$$

that

$$R(X, y) \in \langle 1, X^s, X^{s^2}, \ldots, X^{q/s}, X^q \rangle_{(GF(q))}.$$

In particular this implies that q is a power of s and equivalently that $GF(s)$ is a subfield of $GF(q)$.

If $j \neq q - s^i$ for some i then $\sigma_j(y) = 0$ for all $y \in (GF(q) \setminus D) \bigcup D^\infty$. By comparing the degree of σ_j with the number of zeros of σ_j we see that $\sigma_j \equiv 0$ whenever $j - 1 \leq q - N + N - q/s^2 + 1$.

If there exists a $y \in D \setminus D^\infty$ such that

$$R(X, y) \in GF(q)[X^s] \setminus GF(q)[X^{sp}]$$

then $R(X, y)^{1/s} = X^{q/s} + g(X) \in GF(q)[X] \setminus GF(q)[X^p]$ and by Lemma 1.2 we know that $g^\circ \geq (q + s)/s(s + 1)$. By the previous paragraph if $g^\circ \neq q/s^2$ then $g^\circ < q/s^3$ which contradicts this. Hence $g^\circ = q/s^2$ and $(g')^\circ < q/s^3$. If $s > 2$ then Lemma 1.3 implies that

$$R(X, y) \in \langle X^s, X^{s^2}, \ldots, X^{q/s}, X^q \rangle_{(GF(q))}.$$

5. The graph of f is a subspace

Let $D_u \subset D$ be the subset of elements of D for which $y \in D_u$ implies

$$R(X, y) \in GF(q)[X^u] \setminus GF(q)[X^{up}].$$

Let $t > s$ be minimal such that $D_t \neq \emptyset$.

Counting points of the graph of f on lines incident with a point of the graph gives

$$|\bigcup_{u>s} D_u| \leq \frac{q - 1 - N(s-1)}{t - s} \leq \frac{q - s^2}{s(t - s)}.$$

If $j \neq s^i$ for some i then $\sigma_{q-j}(y) = 0$ for all $y \in GF(q) \setminus (\bigcup_{u>s} D_u)$. By degrees $\sigma_{q-j} \equiv 0$ for $j \geq (q - s^2)/s(t - s)$.

The remainder of the proof is as in [1]. It is included here for the sake of completeness.

Let $y \in D_t$. Write

$$R(X, y) = X^q + g(X) + h(X)^t$$

where $g(X)$ has non-zero terms of degree s^j for some j. We have that

$$(h^t)^\circ < (q-s^2)/s(t-s).$$

If t is not a power of s then $s^i < t < s^{i+1}$ for some i. Extracting t-th roots and following the proof of Lemma 1.2

$$R^{1/t} \mid (X+g+h^t)h'.$$

For $1 \leq j \leq i$ we have

$$X^{q/s^j} + g^{1/s^j} + h^{t/s^j} = 0 \mod R^{1/t}.$$

The polynomial g^{1/s^j} has degree at most q/s^{j+1} and terms of degrees that are powers of s. We can reduce g in the divisibility modulo $R^{1/t}$ to a polynomial in

$$\langle X, X^s, \ldots, X^{q/s^{i+1}}, h^{t/s}, \ldots, h^{t/s^i}\rangle.$$

Note that

$$th^\circ \leq \frac{q-s^2}{s(t-s)} \leq \frac{q}{s^{i+1}}$$

and

$$(h')^\circ < q/s^{i+1}t < q(s^{i+1}-t)/s^{i+1}t < q/t - q/s^{i+1}.$$

So the right-hand side of the divisibility has degree less than q/t after the reduction of g and is therefore zero. However $h' \neq 0$ and so

$$h^t \in \langle X, X^s, \ldots, X^{q/s^{i+1}}, h^{t/s}, \ldots, h^{t/s^i}\rangle.$$

which is impossible since t is not a power of s.

If $t = s^i$ for some $i > 1$ then we go through the same argument replacing h' by $(g^{1/t}+h)'$ and concluding that some linear combination of

$$X, X^s, \ldots, X^{q/s^{i+1}}, h, h^s, \ldots, h^{s^{i-1}}$$

is zero. It follows from this that

$$h \in \langle X, X^s, X^{s^2}, \ldots, X^{q/s^i}\rangle.$$

Hence for all $y \in GF(q)$

$$R(X,y) \in \langle X, X^s, X^{s^2}, \ldots X^q\rangle.$$

Let $w \in GF(q^2) \setminus GF(q)$ and map $AG(2,q)$ to $GF(q^2)$ by $(a,b) \mapsto -aw+b$. The images of the points of the graph of f under this map are zeros of

$$R(X,w) = \prod_{x \in GF(q)} (X + xw - f(x)) \in \langle X, X^s, X^{s^2}, \ldots X^q\rangle_{GF(q^2)}.$$

Now $R(X_1+X_2,w) = R(X_1,w) + R(X_2,w)$ and $R(\alpha X,w) = \alpha R(X,w)$ for all $\alpha \in GF(s)$.

□

6. Further comments

All the bounds in Theorem 1.1 can be attained. The function $f(X) = X^{(q+1)/2}$ determines $(q+3)/2$ directions when q is odd. The function $f(X) = Tr_{q \to s}(X) = X + X^s + X^{s^2} + \ldots + X^{q/s}$ determines $q/s + 1$ directions and the function $f(X) = X^s$ determines $(q-1)/(s-1)$ directions whenever $GF(s)$ is a subfield of $GF(q)$.

The following theorem is from Lovász and Schrijver [5].

Theorem 6.1 *Any function from $GF(q)$ to $GF(q)$ that determines $(q+3)/2$ directions when q is prime is affinely equivalent to $f(X) = X^{(q+1)/2}$.*

The following theorem is from Gács [4].

Theorem 6.2 *Any function from $GF(q)$ to $GF(q)$ that determines more than $(q+3)/2$ directions when q is prime determines at least $\lfloor 2(q-1)/3 \rfloor + 1$ directions.*

There are examples of Megyesi [7] that determine $q + 1 - d$ directions for every d that divides $q - 1$ with $1 < d < q - 1$.

The following is due to Gács, Lovász and Szőnyi [4].

Theorem 6.3 *Let $q = p^2$ where p is prime. Any function from $GF(q)$ to $GF(q)$ that determines $(q+3)/2$ directions is affinely equivalent to $f(X) = X^{(q+1)/2}$. Moreover any function that determines more than $(q+3)/2$ directions determines at least $(q+p)/2 + 1$ directions.*

There are examples of Polverino, Szőnyi and Weiner [6] that determine $(q + \sqrt{q})/2 + 1$ directions when q is square.

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