# Bounds on $(n, r)$-arcs and their application to linear codes

S. Ball* and J.W.P. Hirschfeld

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**Abstract**

This article reviews some of the principal and recently-discovered lower and upper bounds on the maximum size of $(n, r)$-arcs in $\mathrm{PG}(2, q)$, sets of $n$ points with at most $r$ points on a line. Some of the upper bounds are used to improve the Griesmer bound for linear codes in certain cases. Also, a table is included showing the current best upper and lower bounds for $q \le 19$, and a number of open problems are discussed.

## 1 Background

The *weight* of a vector $v$ is the number of non-zero coordinates of $v$. Let $V$ be the $n$-dimensional vector space over $\mathbf{F}_q$. A *linear* $[n, k, d]$*-code* $\mathcal{C}$ *over* $\mathbf{F}_q$ is a $k$-dimensional subspace of $V$ all of whose non-zero vectors have weight at least $d$. Let $v_1, v_2, \dots, v_k$ be a basis for $\mathcal{C}$ and for $i = 1, 2, \dots, n$ define vectors $u_i$ of $V$, by the rule

$$(u_i)_j = (v_j)_i.$$

In other words, the $j$-th co-ordinate of $u_i$ is the $i$-th coordinate of $v_j$. For all $a \in (\mathbf{F}_q)^k$ the vector $\sum_{j=1}^{k} a_j v_j$ has at most $n - d$ zero coordinates and so, for $i = 1, 2, \dots, n$,

$$\sum_{j=1}^{k} a_j (v_j)_i = 0$$

has at most $n - d$ solutions. Hence

$$\sum_{j=1}^{k} a_j (u_i)_j = 0$$

has at most $n - d$ solutions, or in other words there are at most $n - d$ of the $n$ vectors $u_i$ on the hyperplane with equation

$$\sum_{j=1}^{k} a_j X_j = 0.$$

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The matrix whose rows are the vectors $v_i$, or equivalently whose columns are the vectors $u_i$, is called a *generator matrix* of the code $\mathcal{C}$. An $(n, r)$-*arc* in $\mathrm{PG}(k-1, q)$ is a set of $n$ points $\mathcal{K}$ with the property that every hyperplane is incident with at most $r$ points of $\mathcal{K}$ and there is some hyperplane incident with exactly $r$ points of $\mathcal{K}$. Hence an $(n, n-d)$-arc in $\mathrm{PG}(k-1, q)$ is equivalent to a linear $[n, k, d]$-code where $\langle u_i \rangle \neq \langle u_m \rangle$ for $i \neq m$, that is, linear codes for which any two columns of the generator matrix are linearly independent.

The aim of this article is to formulate the bounds on $(n, r)$-arcs as bounds that will look more familiar to coding theorists, to survey recent improvements and list a number of open problems.

For further background to linear codes see [43] or [36], and for $(n, r)$-arcs in $\mathrm{PG}(2, q)$, see [32, Chapter 12]. In various articles and books, when $r$ is large, the complement of a $(n, r)$-arc is considered; this is called a $t$-fold blocking set.

A *$t$-fold blocking set with respect to hyperplanes* is a set of points that is incident with at least $t$ points of every hyperplane and there is some hyperplane incident with exactly $t$ points of the set. In this article it is preferred to leave everything in the language of $(n, r)$-arcs. Note that an $(n, r)$-arc in $\mathrm{PG}(k-1, q)$ is the complement of a $(q+1-r)$-fold blocking set of hyperplanes of size $q^{k-1} + q^{k-2} + \ldots + 1 - n$. An alternative description used is that of an $\{n, m; N, q\}$-*minihyper* which is an $m$-fold blocking set with respect to hyperplanes of $\mathrm{PG}(N, q)$ of size $n$.

## 2 Bounds on $(n, r)$-arcs

In this section, attention is restricted to the case $k = 3$; that is, $(n, r)$-arcs in the plane $\mathrm{PG}(2, q)$ are considered.

Let $\mathcal{K}$ be an $(n, r)$-arc and $P$ be a point of $\mathcal{K}$. Each line incident with $P$ contains at most $r-1$ points of $\mathcal{K} \setminus P$ and the trivial upper bound is obtained:

$$n \leq (r-1)(q+1) + 1 = (r-1)q + r.$$

Cossu [21] noted that when the upper bound is attained every line is incident with either zero or $r$ points of $\mathcal{K}$ and if $r \leq q$, by counting points of $\mathcal{K}$ on lines incident with a point $Q$ not in $\mathcal{K}$, that $r$ divides $q$. In the case when $q$ is even, such arcs exist for every $r$ dividing $q$. In the cases $r = 2$ and $r = q/2$, the arcs are called *hyperovals* and *dual hyperovals* respectively; the known examples are detailed in [34]. There are examples for all $r$ dividing $q$, due to Denniston. Recently, Mathon [41], Mathon and Hamilton [29] and Hamilton [28] constructed many new examples. In the case that $q$ is odd the upper bound can be realised only in the trivial cases $r = 1$, $r = q$ and $r = q + 1$. This was shown in [7]; see [6] for an easier proof. The investigation of $(n, r)$-arcs was initiated by Barlotti [8] whose early work now implies that, if $(r, q) \neq (2^t, 2^h)$ and $2 < r < q$, then

$$n \leq (r-1)q + r - 2.$$

An almost complete table of the known upper bounds can be found in [34, Table 5.2]. The only bound to have been published since then is in the case $r \mid q$ and $q$ odd, where Weiner [46] improved Szőnyi's bound [42],

$$n \leq (r-1)q + r - \tfrac{1}{2}\sqrt[4]{q},$$

to

$$n \leq (r-1)q + r - \tfrac{1}{4}\sqrt{q}$$

for $r \leq \frac{1}{2}\sqrt{q}$. There is one bound that appears in [34, Table 5.2] which is attributed to an unpublished manuscript of the first author. However, the bound is not quite correct as the strictly less than should be a less than or equal to. It is obtained as a corollary to the following theorem, to which a proof is provided since it has not appeared anywhere else.

**Theorem 2.1.** *If there exists an $((r-1)q+\epsilon, r)$-arc $\mathcal{K}$ with $\epsilon \geq 1$ in a projective plane $\pi$ of order $q$ which has no skew line, then*

$$r^2 - \epsilon q + \epsilon(\epsilon - r) - r \geq 0.$$

**Proof** Let $n = (r-1)q + \epsilon$. Counting points of $\mathcal{K}$ on each line through a point $P$ of $\mathcal{K}$, it is seen that every line meets $\mathcal{K}$ in at least $\epsilon$ points. Bruen's idea [17] is extended to look at the inequality,

$$\sum_{i=\epsilon}^{r} (r-i)(i-\epsilon)\tau_i \geq 0,$$

where $\tau_i$ is the number of lines meeting $\mathcal{K}$ in $i$ points. Standard counting arguments for a point set in a projective plane give

$$\sum_{i=\epsilon}^{r} \tau_i = q^2 + q + 1, \quad \sum_{i=\epsilon}^{r} i\tau_i = n(q+1), \quad \sum_{i=\epsilon}^{r} i(i-1)\tau_i = n(n-1),$$

and, combining these with the inequality, implies that

$$-n(n-1) + (\epsilon + r - 1)n(q+1) - \epsilon r(q^2+q+1) \geq 0.$$

By calculation this gives

$$r^2 - \epsilon q + \epsilon(\epsilon - r) - r \geq 0.$$

□

**Corollary 2.2.** *An $(n, r)$-arc $\mathcal{K}$ in a projective plane $\pi$ of order $q$ which has no skew line satisfies*

$$n \leq (r-1)q + \left\lfloor \frac{r^2}{q} \right\rfloor,$$

*and if $\sqrt{q}$ divides $r$ then*

$$n < (r-1)q + \left\lfloor \frac{r^2}{q} \right\rfloor.$$

**Proof** Theorem 2.1 provides a contradiction for $\epsilon \geq r^2/q$. □

Corollary 2.2 in combination with the following from [4] can always be used to provide an upper bound.

An $(n, r)$-arc $\mathcal{K}$ in a projective plane $\mathrm{PG}(2, q)$ which has a skew line satisfies

$$n \leq (r-1)q + p^e,$$

where $(r, q) = p^e$.

Table A lists all $r$ for which there is known to exist an $(n, r)$-arc with$n > (r-2)q + r$ and the maximum value of $n$ known in that case. An $(n, r)$-arc in PG$(2, q)$, with $n > (r-2)q + r$, is equivalent to a code meeting the Griesmer bound, see Section 3.

In the table, the integer $q = p^h$ is *exceptional* if $h$ is odd, $h \geq 3$ and $p$ divides $\lfloor 2\sqrt{q} \rfloor$.

| $r$ | $q$ | $n$ | |
|---|---|---|---|
| 2 | $q$ odd | $(r-1)q+1$ | [13] |
| $2^e$ | $2^h$ | $(r-1)q+r$ | [26] |
| $r=3$ | $q$ non-exceptional | $(r-2)q+1+\lfloor 2\sqrt{q} \rfloor$ | [45] |
| $r=3$ | $q$ exceptional | $(r-2)q+\lfloor 2\sqrt{q} \rfloor$ | [45] |
| $r=\sqrt{q}+1$ | $q$ square | $(r-1)q+1$ | [14] |
| $r=(q+1)/2$ | $q$ odd | $(r-1)q+1$ | [9] |
| $r=(q+3)/2$ | $q$ odd | $(r-1)q+1$ | [9] |
| $r=q-\sqrt{q}$ | $q$ square | $(r-1)q+\sqrt{q}$ | [39] |
| $r=q-p^e$ | $q=p^h$ | $(r-1)q+q-r$ | [40] |
| $r \geq q+2-\sqrt{q}$, | $q$ square | $(r-1)q+r-\sqrt{q}(q+1-r)$ | [31] |
| $r=q-2$ | $q$ even | $(r-1)q+2$ | [31] |
| $r=q-2$ | $q$ odd | $(r-1)q+1$ | [31] |
| $r=q-1$ | $q$ | $(r-1)q+1$ | [31] |

Table A : The known families of $(n, r)$-arcs in PG$(2, q)$ with $n > (r-2)q + r$

Large arcs can also be constructed from the set of rational points of an algebraic curve, sometimes by adding extra points; see Daskalov and Jiménez Contreras [24], Giulietti *et. al.* [27] and Voloch [44].

# 3 Bounds on linear codes

In this section, some of the upper bounds on $(n, r)$-arcs are reformulated in terms of linear codes. This gives a Griesmer-like bound (3.1) for three-dimensional codes which is essentially nothing new but only novel in its formulation. Corollary 3.2 generalises the bound to higher-dimensional codes.

Recall that for a linear $[n, k, d]$-code the Griesmer bound, [43, Theorem 5.2.6], states that

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil .$$

An $(n, r)$-arc in PG$(2, q)$ is equivalent to a linear $[n, 3, n-r]$-code and so the Griesmer bound tells us, assuming $d \leq q^2$,

$$n \geq \sum_{i=0}^{2} \left\lceil \frac{n-r}{q^i} \right\rceil = n - r + \left\lceil \frac{n-r}{q} \right\rceil + 1.$$

Hence we have the upper bound

$$n \leq (r-1)q + r$$

and equality in the Griesmer bound if and only if $n > (r-2)q + r$.

An $(n,r)$-arc in $\mathrm{PG}(2,q)$ satisfies the upper bound,

$$n \leq (r-1)q + 1,$$

in the cases when

(i) (Blokhuis, see [2]) $q$ is prime and $r \leq (q+3)/2$;

(ii) (Section 2) $(r,q) = 1$ and $r < \sqrt{2q} + 1$;

(iii) (Blokhuis [12]) $(r,q) = 1$ and there is a line skew from the $(n,r)$-arc;

(iv) (Weiner [46]) $q$ is odd, $r \mid q$ and $r < \frac{1}{4}\sqrt{q}$.

The following theorem reformulates this bound in terms of linear codes. Put $q = p^h$, where $p$ is prime.

**Theorem 3.1.** *Suppose that one of the following holds:*

(i) *$q$ is prime and $d \leq (q-1)(q+3)/2$;*

(ii) *$\lceil d/(q-1) \rceil \neq -1 \pmod p$ and $d \leq \sqrt{2q}(q-1)$;*

(iii) *$\lceil d/(q-1) \rceil \neq -1 \pmod p$, $d \leq q^2$ and there is a codeword of weight $n$;*

(iv) *$q$ is odd, $n = d + p^e$ for some $e$ and $d < (\frac{1}{4}\sqrt{q} - 1)(q-1)$.*

*Then a linear $[n,3,d]$ code over $\mathbf{F}_q$ satisfies*

$$n \geq d + \left\lceil \frac{d}{q-1} \right\rceil + \left\lceil \frac{d}{q^2} \right\rceil. \tag{3.1}$$

**Proof** Let $\mathcal{C}$ be a linear $[n,3,d]$-code over $\mathbf{F}_q$.

If $\mathcal{C}$ has repeated columns in its generator matrix, it may be assumed that the first two columns are $(1,0,0)^t$. The matrix obtained by deleting the first two columns and the first row generates an $[n-2,2,d]$ linear code. Applying the Griesmer bound,

$$n - 2 \geq d + \left\lceil \frac{d}{q} \right\rceil,$$

from which the bound (3.1) follows since $d \leq q^2$.

If $\mathcal{C}$ has no repeated columns, then an $(n, n-d)$-arc in $\mathrm{PG}(k-1,q)$ is obtained. By assumption,

$$n \leq (n-d-1)q + 1,$$

and so

$$n(q-1) \geq d + d(q-1) + q - 1.$$

Now, dividing by $q-1$ gives the bound (3.1) for $d \leq q^2$.

(i) The condition $r \leq (q+3)/2$ translates to $n \leq d+(q+3)/2$. Hence either the bound (3.1) holds or $n \geq d+(q+5)/2$, which is a better bound if $d \leq (q-1)(q+3)/2$.

(ii) The condition $r < \sqrt{2q}+1$ translates to $n < \sqrt{2q}+1+d$. Hence either the bound (3.1) holds or $n \geq \sqrt{2q}+1+d$ which is a better bound if $d \leq (q-1)\sqrt{2q}$. If equality in the bound violates the condition $(n-d, q) = 1$, then $\lceil d/(q-1) \rceil \equiv -1 \pmod{p}$.

(iii) The condition that there is a line skew from the $(n, r)$-arc translates to the condition that there is a codeword of weight $n$.

(iv) The condition $r \mid q$ translates to $n = d + p^e$ for some $e$. The bound (3.1) holds or $n \geq d + \frac{1}{4}\sqrt{q}$, which is a better bound if $d \leq (q-1)(\frac{1}{4}\sqrt{q}-1)$.

□

**Corollary 3.2.** *Suppose one of the following holds:*

(i) *$q$ is prime and $d \leq q^{k-3}(q-1)(q+3)/2$;*

(ii) *$\lceil d/\{(q-1)q^{k-3}\} \rceil \not\equiv q-1 \pmod{p}$ and $d \leq \sqrt{2q}(q-1)q^{k-3}$;*

(iii) *$\lceil d/\{(q-1)q^{k-3}\} \rceil \not\equiv q-1 \pmod{p}$, $d \leq q^{k-1}$ and there is a codeword of weight $n$;*

(iv) *$q$ is odd, $n = \sum_{i=0}^{k-3} \lceil d/q^i \rceil + p^e$ for some $e$ and $d < (\frac{1}{4}\sqrt{q}-1)(q-1)q^{k-3}$.*

*Then a linear $[n, k, d]$ code over $\mathbf{F}_q$ satisfies*

$$n \geq \sum_{i=0}^{k-3} \left\lceil \frac{d}{q^i} \right\rceil + \left\lceil \frac{d}{(q-1)q^{k-3}} \right\rceil + \left\lceil \frac{d}{q^{k-1}} \right\rceil . \tag{3.2}$$

**Proof** Let $k \geq 4$ and let $\mathcal{C}$ be a linear $[n, k, d]$-code over $\mathbf{F}_q$.

If $\mathcal{C}$ has repeated columns in its generator matrix, assume that the first two columns are $(1, 0, \ldots, 0)^t$. The matrix obtained by deleting the first two columns and the first row generates an $[n-2, k-1, d]$ linear code. Applying the Griesmer bound,

$$n-2 \geq \sum_{i=0}^{k-2} \left\lceil \frac{d}{q^i} \right\rceil ,$$

from which the bound (3.1) follows since $d \leq q^{k-1}$.

If $\mathcal{C}$ has no repeated columns in its generator matrix, then let $\mathcal{K}$ be the corresponding $(n, n-d)$ arc in $\mathrm{PG}(k-1, q)$. There is a hyperplane $H$ meeting $\mathcal{K}$ in $n-d$ points or else $\mathcal{C}$ would have minimum distance more than $d$. Let $e$ be the minimum such that $H \cap \mathcal{K}$ is an $(n-d, n-d-e)$-arc in $\mathrm{PG}(k-2, q)$ and let $L$ be a hyperplane meeting $H \cap \mathcal{K}$ in $n-d-e$ points. Then counting points of $\mathcal{K}$ on hyperplanes containing $L$ gives

$$n \leq eq + n - d,$$

and hence $e \geq \lceil d/q \rceil$. Thus $H \cap \mathcal{K}$ gives us an $[n-d, k-1, \lceil d/q \rceil]$ linear code.

By iteration an $[n-d-\lceil d/q \rceil - \ldots - \lceil d/q^{k-4} \rceil, 3, \lceil d/q^{k-3} \rceil]$ linear code is obtained. Now, according to the conditions, Theorem 3.1 can be applied. □

## 4 Large $(n, r)$-arcs in small planes

Table B is an update of [34, Table 5.4] including results of Daskalov and Medotieva [22, 25] and the many new constructions of Braun *et al.* [16].

The new $(n, r)$-arcs from [16] were found in the following way. Let $M$ be the point-line incidence matrix of $\mathrm{PG}(2, q)$. Then an $(n, r)$-arc is given by a vector $x \in \{0, 1\}^{q^2+q+1}$ with the property that $x$ has $n$ coordinates equal to 1 and the coordinates of $Mx$ are at most $r$. Even for small $q$ the computation involved in solving these equations is unfeasible. For this reason the authors of [16] choose a subgroup $G$ of the automorphism group of $\mathrm{PG}(2, q)$ and consider the reduced incidence matrix $M^G$ whose columns are the $G$-orbits on the points and whose rows are the $G$-orbits on the lines. If $t$ is the number of orbits and $w_j$ is the size of orbit of points corresponding to the $j$-th column of $M^G$, an $(n, r)$-arc is given by a vector $x \in \{0, 1\}^t$ with the property that $\sum_{j=1}^{t} x_j w_j = n$ and the coordinates of $M^G x$ are at most $r$. Various small subgroups $G$ are used, the corresponding systems of diophantine equations are solved exhaustively by computer using lattice-point enumeration.

Table C provides all the references for the lower and upper bounds in Table B. The reference [$*$] means that the bound can be deduced from the bounds in Section 2. In the case of a $(98, 7)$-arc in $\mathrm{PG}(2, 16)$, a $(165, 11)$-arc in $\mathrm{PG}(2, 16)$, a $(182, 12)$-arc in $\mathrm{PG}(2, 16)$, there is equality in the bound of Theorem 2.1. This implies that such arcs, if they have no skew line, have only $\epsilon$-secants and $r$-secants. Easy counting arguments give a contradiction. If they have a skew line the bound from [4] can be used.

The reference [$**$] refers to the following slight improvement of [2, Theorem 4.2]. If there is equality in the bound and $r \geq (q + 1)/2$ then the number of $r$-secants can be counted, following the same arguments as used for the small planes in [5], and is $(q - 1)(2q + 3 - 2r)(q + 1)/2(q + 1 - r)$. If this number is not an integer then the bound can be improved to

$$n \leq (r - 1)q + r - (q + 3)/2.$$

| $q$ <br> $r$ | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 13 | 16 | 17 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 4 | 6 | 6 | 8 | 10 | 10 | 12 | 14 | 18 | 18 | 20 |
| 3 | | 9 | 11 | 15 | 15 | 17 | 21 | 23 | 28...33 | 28...35 | 31...39 |
| 4 | | | 16 | 22 | 28 | 28 | 32...34 | 38...40 | 52 | 48...52 | 52...58 |
| 5 | | | | 29 | 33 | 37 | 43...45 | 49...53 | 65 | 61...69 | 68...77 |
| 6 | | | | 36 | 42 | 48 | 56 | 64...66 | 78...82 | 78...86 | 86...96 |
| 7 | | | | | 49 | 55 | 67 | 79 | 93...97 | 94...103 | 105...115 |
| 8 | | | | | | 65 | 77...78 | 92 | 120 | 114...120 | 124...134 |
| 9 | | | | | | | 89...90 | 105 | 128...131 | 137 | 147...153 |
| 10 | | | | | | | 100...102 | 118...119 | 142...148 | 154 | 172 |
| 11 | | | | | | | | 132...133 | 159...164 | 166...171 | 191 |
| 12 | | | | | | | | 145...147 | 180...181 | 182...189 | 204...210 |
| 13 | | | | | | | | | 195...199 | 204...207 | 225...230 |
| 14 | | | | | | | | | 210...214 | 221...225 | 242...250 |
| 15 | | | | | | | | | 231 | 239...243 | 262...270 |
| 16 | | | | | | | | | | 256...261 | 285...290 |
| 17 | | | | | | | | | | | 305...310 |
| 18 | | | | | | | | | | | 324...330 |

Table B : The size of the largest $(n, r)$-arc in $\mathrm{PG}(2, q)$ for small $q$

| $q$ <br> $r$ | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 13 | 16 | 17 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | [13] | [13] | [13] | [13] | [13] | [13] | [13] | [13] | [13] | [13] | [13] |
| 3 | | [35] | [35] | [15] | [11] | [10] | [33]...[37] | [2]...[38] | [16]...[8] | [16]...[8] | [16]...[8] |
| 4 | | | [35] | [35] | [21] | [35] | [16]...[35] | [16]...[35] | [26] | [16]...[35] | [16]...[35] |
| 5 | | | | [20] | [39] | [39] | [2]...[2] | [16]...[2] | [14]...[$*$] | [16]...[2] | [16]...[2] |
| 6 | | | | [20] | [31] | [39] | [9]...[2] | [16]...[2] | [16]...[$*$] | [16]...[2] | [16]...[2] |
| 7 | | | | | [11] | [3] | [9]...[2] | [9]...[2] | [16]...[$*$] | [16]...[2] | [16]...[2] |
| 8 | | | | | | [5] | [2]...[2] | [9]...[2] | [26] | [16]...[2] | [16]...[2] |
| 9 | | | | | | | [31]...[5] | [2]...[2] | [16]...[$*$] | [9]...[2] | [16]...[2] |
| 10 | | | | | | | [31]...[5] | [16]...[2] | [16]...[$*$] | [9]...[2] | [9]...[2] |
| 11 | | | | | | | | [16]...[3] | [16]...[$*$] | [16]...[22] | [9]...[2] |
| 12 | | | | | | | | [16]...[5] | [16]...[$*$] | [22]...[22] | [16]...[$**$] |
| 13 | | | | | | | | | [16]...[30] | [16]...[22] | [16]...[$**$] |
| 14 | | | | | | | | | [31]...[3] | [25]...[22] | [16]...[$**$] |
| 15 | | | | | | | | | [5] | [31]...[22] | [16]...[23] |
| 16 | | | | | | | | | | [31]...[5] | [16]...[$**$] |
| 17 | | | | | | | | | | | [31]...[23] |
| 18 | | | | | | | | | | | [31]...[5] |

Table C : References for the size of the largest $(n, r)$-arc in $\mathrm{PG}(2, q)$ for small $q$

# 5 Open problems

1. In most cases no example is known of an $(n, r)$-arc with $n/q$ large, say $n/q > r-2$. The best that can be done in general is to take (a) for $r < q/2$ the union of $\lfloor r/2 \rfloor$ conics, which

gives $n/q > r/2$ and (b) for $r > q/2$ large the complement of the union of $2(q-r)+1$ lines of a dual $(2(q-r)+1,2)$-arc, which gives $n/q > q-2r+(2r^2-r)/q$.

2. In the case $r=3$ there is a construction of size $q+\lfloor 2\sqrt{q}\rfloor$ for all $q$ and an upper bound of $n \leq 2q+1$. Any progress on determining a constant $c$ such that the upper bound $n/q < c < 2$ for $q$ large enough, or a construction where $n/q > c > 1$ for infinitely many $q$ will be rewarded by a cheque for 10,000 Hungarian florins from Prof. A. Blokhuis.

3. In the case $r = q-1$ Braun *et al.*'s [16] discovery of a (145,12)-arc in $\mathrm{PG}(2,13)$ ends speculation that an $(n,q-1)$-arc in $\mathrm{PG}(2,q)$, $q$ prime, satisfies $n \leq (q-2)q+1$; the so-called $3p$ *conjecture for double blocking sets*, see [5]. It is known from [5] that $n \leq (q-2)q+\frac{q-3}{2}$ but in general there is no better construction than the complement of three non-concurrent lines, which provides an example with $n=(q-2)q+1$. Any construction of a family of $(n,q-1)$-arcs in $\mathrm{PG}(2,q)$, for infinitely many $q$ prime, with $n \geq (q-2)q+2$ would be of interest.

4. For $q$ prime, there are upper bounds on $n$ due to Blokhuis which appear in [2]. For $q$ non-prime and $r > \sqrt{q}+1$, there are few upper bounds on $n$ that use the fact that the projective plane is Desarguesian; in other words only counting arguments are used. The only exceptions are when $r > q-q^{1/6}$ and $q$ is square.

## References


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S. Ball
Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya
Jordi Girona 1-3
Mòdul C3, Campus Nord
08034 Barcelona
Spain
`simeon@mat.upc.es`
`http://www-ma4.upc.es/~simeon/`

J.W.P. Hirschfeld
Department of Mathematics
University of Sussex
Brighton BN1 9RF
United Kingdom
`jwph@sussex.ac.uk`
`http://www.maths.susx.ac.uk/Staff/JWPH/`