On nuclei and blocking sets in Desarguesian spaces

Simeon Ball *

Technische Universiteit Eindhoven,
5600 MB Eindhoven,
The Netherlands.

Abstract

A generalisation is given to recent results concerning the possible number of nuclei to a set of points in $PG(n,q)$. As an application of this we obtain new lower bounds on the size of a t-fold blocking set of $AG(n,q)$ in the case $(t,q)>1$.

1. Introduction

Let $\Pi_n = PG(n,q)$ be the n-dimensional Desarguesian projective space of order q and let $\mathcal{A}_n = AG(n,q)$ be the n-dimensional Desarguesian affine space of order q. Let $\mathcal{S}$ be a subset of the points of Π_n. A point P of $\Pi_n \setminus \mathcal{S}$ is a *t-fold nucleus* of $\mathcal{S}$ if every line through P meets at least t points of $\mathcal{S}$. This definition is consistent with [8] where some background results concerning nuclei appear for $n=2$ and $t=1$ in Section 13.7, and general t in Section 13.8. For general n the definition follows Sziklai [10]. Let $\mathcal{N}^t(\mathcal{S})$ denote the set of t-fold nuclei of $\mathcal{S}$. A 1-fold nucleus is called a nucleus. Recall that a set $\mathcal{S}$ is a *t-fold blocking set* if every line contains at least t points of $\mathcal{S}$ or, equivalently, the set $\mathcal{S} \cup \mathcal{N}^t(\mathcal{S})$ contains every point. A 1-fold blocking set is a blocking set. For a detailed background on blocking sets, multiple blocking sets and nuclei in $\mathcal{A}_2$ and Π_2, see [8].

The main result of [1] is that a $(q+1)$-set in $\mathcal{A}_2$ has at most $q-1$ nuclei. The only known examples of sets that have this number of nuclei are a set consisting of a line together with a point, and a sporadic example in $AG(2,5)$, where the 10 points of a Desargues configuration can be partitioned into a set of size 6 and 4 nuclei. It remains an intriguing problem to characterise the sets $\mathcal{S}$ of size $q+1$ having exactly $q-1$ nuclei. Partial results in this direction appear in [2].

In [3] it is shown that a $(q+k)$-set in $\mathcal{A}_2$ has at most $k(q-1)$ nuclei, and as a consequence of this that a blocking set in $\mathcal{A}_2$ has at least $2q-1$ points, a result proved originally by Jamison [9] and independently Brouwer and Schrijver [5]. This bound was generalised to $tq+q-t$ for t-fold blocking sets in $\mathcal{A}_2$ by Bruen [6].

The notion of t-fold nucleus was first introduced by Blokhuis in [4], where it was shown that a $(tq+t+k-1)$-set has at most $k(q-1)$ t-fold nuclei provided that the binomial coefficient $\binom{t+k-1}{k}$ is non-zero modulo p, where p denotes the characteristic of $GF(q)$. It follows directly from this that a t-fold blocking set in $\mathcal{A}_2$ has at least $tq+q-1$ points

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for $(t,q)=1$. Blokhuis mentions that it is surprising that his nuclei result implies lower bounds for t-fold blocking sets in $\mathcal{A}_2$ for $(t,q)=1$ yet does not appear to imply Bruen's general bound. He states "It seemed therefore natural to expect that this [Bruen's] bound also would follow as a corollary from a more general result on multiple nuclei". In this paper such a general result is given, however, Bruen's bound and the new lower bounds for blocking sets in $\mathcal{A}_2$ follow directly from Blokhuis' result as well. To conclude this introduction we mention that Sziklai [10] has generalised the result in [4]. He concludes that a set $\mathcal{S}$ of size $t\theta_{n-1}+k-1$, with i points lying in some hyperplane $\mathcal{H}$, has at most $k(q-1)$ t-fold nuclei in $\mathcal{A}_n=\Pi_n\setminus\mathcal{H}$, provided that the binomial coefficient $\binom{t\theta_{n-1}+k-i-1}{k}$ is non-zero modulo p. Throughout $\theta_n=(q^{n+1}-1)/(q-1)$, the number of points of Π_n.

2. Multiple nuclei

Let $\mathcal{S}$ be a subset of $t\theta_{n-1}+k-1$ points of Π_n. The points of $\mathcal{A}_n$ can be identified with the elements of $GF(q^n)$ in a suitable way, so that in fact all point sets can be considered as subsets of this field. Note that three points a, b, c are collinear, precisely when $(a-b)^{q-1}=(a-c)^{q-1}$. If the direction of the line joining a and b is identified with the number $(a-b)^{q-1}$, then a one-to-one correspondence between the θ_{n-1} directions (or parallel classes) and the different θ_{n-1}-st roots of unity in $GF(q^n)$ is obtained.

Theorem 2.1 *Suppose there exists a hyperplane $\mathcal{H}$ containing exactly i points of $\mathcal{S}$. The number of t-fold nuclei in $\mathcal{A}_n=\Pi_n\setminus\mathcal{H}$ is at most $(k+r)(q-1)$ provided that the binomial coefficient*

$$\binom{t\theta_{n-1}+k-i-1}{k+r}\neq 0 \pmod{p},$$

for some $r\geq 0$.

Proof : We restrict ourselves to the case $(k+r)(q-1)<q^n-(|\mathcal{S}|-i)$, which implies $k+r<\theta_{n-1}$, since otherwise the bound is obvious. Let $\mathcal{S}_{\mathcal{H}}(=\mathcal{S}\setminus\mathcal{H})$ be the points of $\mathcal{S}$ contained in the affine space $\mathcal{A}_n=\Pi_n\setminus\mathcal{H}$ and consider $\mathcal{S}_{\mathcal{H}}$ as a subset of $GF(q^n)$. We assume without loss of generality that 0 is not in $\mathcal{S}$ and is not a t-fold nucleus of $\mathcal{S}$. Let $\delta_1,....,\delta_i\in GF(q^n)$ correspond to the directions of the lines meeting at the i points of $\mathcal{S}$ in $\mathcal{H}$. Note that since the δ_m correspond to directions $\delta_m^{\theta_{n-1}}=1$ or, alternatively, δ_m is a non-zero $(q-1)$-st power.

Define polynomials F in two variables and σ_j in one variable by

$$F(u,x)=\prod_{m=1}^{i}(1-\delta_m x^{q-1}u)\prod_{b\in\mathcal{S}_{\mathcal{H}}}(1-(1-bx)^{q-1}u)=\sum_{j=0}^{t\theta_{n-1}+k-1}(-1)^j\sigma_j(x)u^j,$$

where σ_j is the j-th symmetric function of the set

$$\mathcal{U}=\{\delta_m x^{q-1}|m=1,...,i\}\cup\{(1-bx)^{q-1}|b\in\mathcal{S}_{\mathcal{H}}\}.$$

Note that the degree of the polynomial $\sigma_j(x)$ is at most $j(q-1)$. Let $N^t_{\mathcal{H}}(\mathcal{S})$ denote the set of t-fold nuclei of the set $\mathcal{S}$ contained in the affine space $\Pi_n\setminus\mathcal{H}$ and define $N^t_{\mathcal{H}}(\mathcal{S})^{[-1]}=\{1/x_0|x_0\in N^t_{\mathcal{H}}(\mathcal{S})\}$.

Consider each linear factor of $F(u,x_0)$ where $x_0\in GF(q^n)$. Each one corresponds to the direction of the line joining $1/x_0$ to a point of $\mathcal{S}$. Note that $(1-bx)^{q-1}=x^{q-1}(1/x-b)^{q-1}$ and in each linear term the direction is permuted by x_0^{q-1}.

For every $x_0 \in N^t_{\mathcal{H}}(\mathcal{S})^{[-1]}$ the multi-set $\mathcal{U}$ contains every θ_{n-1} direction (alternatively θ_{n-1}-st roots of unity) repeated at least t times since there are at least t points of $\mathcal{S}$ on every line through the point corresponding to $1/x_0$. Hence

$$F(u,x_0) = (1-u^{\theta_{n-1}})^t.G(u,x_0) = \sum_{j=0}^{t\theta_{n-1}+k-1} (-1)^j \sigma_j(x_0) u^j$$

where $G(u,x_0)$ is a polynomial of degree $k-1$ in u. The shape of $F(u,x_0)$ implies that the polynomial σ_{k+r} is zero for all $x_0 \in \mathcal{N}^t_{\mathcal{H}}(\mathcal{S})^{[-1]}$ under the assumption that $k+r < \theta_{n-1}$. Hence if $|\mathcal{N}^t_{\mathcal{H}}(\mathcal{S})| > (k+r)(q-1)$ then $\sigma_{k+r} \equiv 0$, since it has degree at most $(k+r)(q-1)$ by definition.

However

$$F(u,0) = (1-u)^{t\theta_{n-1}+k-i-1}$$

and so $\sigma_{k+r}(0)$ is non-zero, and hence σ_{k+r} is not identically zero, whenever

$$\binom{t\theta_{n-1}+k-i-1}{k+r} \neq 0 \pmod{p}.$$

□

3. Affine blocking sets

Recall that a t-fold blocking set in $\mathcal{A}_n$ is a subset $\mathcal{S}$ of points meeting every line of $\mathcal{A}_n$ at least t times. An application of the previous theorem with $i = r = 0$ allows us to prove the following theorem. For $(t,q) = 1$ this bound can be found in Sziklai [10].

Theorem 3.1 *Let $\mathcal{S}$ be a t-fold blocking set of $\mathcal{A}_n$ and let $e(t)$ be maximal such that $p^{e(t)}|t$. Then the set $\mathcal{S}$ has at least $(t+1)q^{n-1} - p^{e(t)}$ points.*

Proof : Put $k = q^{n-1} - t\theta_{n-2} - p^{e(t)}$ and write $t = \gamma p^{e(t)}$ such that $p \not| \gamma$. Consider the binomial coefficient

$$\binom{t\theta_{n-1}+k-1}{k} = \binom{t\theta_{n-1}+k-1}{t\theta_{n-1}-1} = \binom{tq^{n-1}+q(q^{n-2}-1)+q-p^{e(t)}-1}{tq^{n-1}+tq^{n-2}+....+tq+t-1}.$$

A simple application of Lucas' Theorem implies that this binomial coefficient is non-zero precisely when

$$\binom{q-p^{e(t)}-1}{\gamma p^{e(t)}-1} = \binom{q-2p^{e(t)}+p^{e(t)}-1}{(\gamma-1)p^{e(t)}+p^{e(t)}-1} = \binom{q/p^{e(t)}-2}{\gamma-1} \pmod{p}$$

is non-zero, and it is non-zero since $\gamma \neq 0 \bmod p$. Hence $\mathcal{S}$ cannot be a t-fold blocking set when $k = q^{n-1} - t\theta_{n-2} - p^{e(t)}$ since $\mathcal{S}$ has at most $k(q-1)$ t-fold nuclei and

$$t\theta_{n-1} + k - 1 + k(q-1) = t\theta_{n-1} + kq - 1 =$$

$$(q^{n-1} - t\theta_{n-2} - p^{e(t)})q + t\theta_{n-1} - 1 = q^n + t - p^{e(t)}q - 1 < q^n.$$

□

Hence for $n = 2$ we have proved both the previous lower bounds due to Bruen and Blokhuis. It is worth noting however that we have used Theorem 2.1 only in the case $i = r = 0$ which for $n = 2$ is precisely Blokhuis' result on multiple nuclei. The new lower bounds occur for all t such that $(t, q) > 1$ and $t \neq p^e$ for some e. Finally we note that in $AG(2, 2^h)$ the bound is sharp when $t = q - 2^e$. Consider a $(q^2 - 2^e q + q - 2^e)$-set $\mathcal{S}$, the complement of which is a $(2^e q - q + 2^e, 2^e)$-(maximal) arc. A construction due to Denniston [7] implies the existence of such sets for all e. It follows immediately that $\mathcal{S}$ is a $(q - 2^e)$-fold blocking set of $\mathcal{A}_2$ attaining the bound.

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