TWO-PARAMETER BIFURCATION CURVES IN POWER ELECTRONIC CONVERTERS

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In this paper we prove the general result that, given a linear system \( \dot{x} = Ax + u \) where \( A \) is hyperbolic, \( u \) is piecewise linear and \( L \)-periodic, with \( \int_0^L u(t)dt = 0 \), then there exists a unique \( L \)-periodic solution \( x = x_p(t) \) such that \( \int_0^L x_p(t)dt = 0 \). We then consider a DC/DC buck (step-down) converter controlled by the ZAD (zero-average dynamics) strategy. The ZAD strategy sets the duty cycle, \( d \) (the length of time the input voltage is applied across an inductance), by ensuring that, on average, a function of the state variables is always zero. The two control parameters are \( v_{ref} \), a reference voltage that the circuit is required to follow, and \( k_s \), a time constant which controls the approach to the zero average. We show how to calculate \( d \) exactly for a periodic system response, without knowledge of the state space solutions. In particular, we show that for a \( T \)-periodic response \( d \) is independent of \( k_s \). We calculate period doubling and corner collision bifurcations, the latter occurring when the duty cycle saturates and is unable to switch. We also show the presence of a codimension two nonsmooth bifurcation in this system when a corner collision bifurcation and a saddle node bifurcation collide.

Keywords: Averaging method; nonsmooth bifurcation; DC/DC converters; control theory.

1. Introduction
Electronic devices such as mobile phones and laptop computers contain circuits that require differing voltage levels. The voltage is changed by switched mode conversion in a circuit known as a power electronic converter. The input voltage is applied across an inductance, then switched and the stored energy transferred to the output voltage. An account of power electronic converter is given in [Mohan et al., 2002]. But the output of power electronic converters can be chaotic [Fossas & Olivar, 1996; Yuan et al., 1998]. Therefore, some form of control strategy must be introduced. Banerjee and Verghese [2001] gave a detailed account of the modeling of power electronic converters by ordinary differential equations with a discontinuous right-hand side. Di Bernardo et al. [2007] gave a comprehensive account.
of the latest theoretical developments in the study of such equations.

In this paper, we shall study the example of the DC/DC buck (step-down) converter controlled by the ZAD (zero-average dynamics) strategy [Biel et al., 2001]. In Sec. 2 we introduce the DC/DC buck converter with ZAD strategy. In Sec. 3, we calculate exactly the duty cycle of a periodic system response, in the absence of detailed knowledge of the dynamics of the problem. In Sec. 4, we analytically calculate the period doubling bifurcation curve of a 2-periodic system solution in $(v_{ref}, k_0)$ parameter space. We perform a similar calculation in Sec. 5, where we find the corner collision bifurcation curve in $(v_{ref}, k_1)$ space where the 2T-periodic solution saturates (that is, when one duty cycle of the response equals 0 or 1). In Sec. 6 we show that the period doubling bifurcation curve and the corner collision bifurcation curve intersect tangentially, normal to and at the line $v_{ref} = 1$ and tangential at the line $v_{ref} = 0$. We also note in this section a codimension two nonsmooth bifurcation point. Finally in Sec. 7, we provide three different codimension one bifurcation diagrams that occur in this system as the duty cycle, $d$, varies with $k_0$ for fixed $v_{ref}$.

2. The DC/DC Buck Converter with ZAD Strategy

The buck converter circuit switches between two distinct linear topologies depending on the value of a control input. The control action considered here, zero average dynamics (ZAD), was proposed in [Fossas et al., 2001] and [Ramos et al., 2003]. ZAD as implemented here is effectively an averaged PID controller. The benefits of the ZAD strategy include simplicity and practicality. The ZAD controller yields a converter performance more robust than those usually obtained by PID controllers, specifically with respect to nonlinear loads. It involves the direct design of the duty cycle and is implemented in a single updated lateral pulse width modulation (PWM). Figure 1 depicts the block diagram of the converter. Other topologies are possible. The signal reference $u_{ref}$ corresponds to the required output voltage which we shall take to be constant.

The control input $u$ takes discrete, constant, values in the set $\{B_1, B_2\}$. In this paper, we examine the case $B_1 = 1$, $B_2 = 0$, corresponding to the lateral PWM case. The voltage $w$ on the capacitor and the current $i$ on the inductor are the dependent (state) variables. We apply the following non-dimensionalization: $x_1 = w/E$, $x_2 = (1/E)\sqrt{(TE/LC)}$, $d = \tau/\sqrt{LC}$ and define parameters $\gamma = (1/R)\sqrt{(L/C)}$ and $T = T_c/\sqrt{LC}$. In addition we set $v_{ref} = u_{ref}/E$.

Parameter values are chosen as $R = 209$, $C = 40\mu F$, $L = 2\h$, $E = 40\, \text{V}$. The clock cycle (sampling period) is $T_c = 50\, \mu s$, hence $\gamma = 0.35$ and $T = 0.1767$. Then the equations of the DC/DC buck converter in dimensionless form can be written as

$$\dot{x} = Ax + u$$

where $x = (x_1, x_2)$, $u$ is given by

$$u = \begin{cases} B_1, & kT \leq t < kT + \frac{Td_h}{2} \\ B_2, & kT + \frac{Td_h}{2} \leq t < (k+1)T - \frac{Td_h}{2} \\ B_1, & (k+1)T - \frac{Td_h}{2} \leq t < (k+1)T \end{cases}$$

where $d_h$ is the (scaled) duty cycle and

$$A = \begin{pmatrix} -\gamma & 1 \\ -1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note that $d_h \in [0, 1]$ and that $d_h = 0$ corresponds to $u = B_2$ and $d_h = 1$ corresponds to $u = B_1$ on $[kT, (k+1)T])$.

In (2), the ZAD strategy determines the duty cycle $d_h$ and guarantees that the output $x_1$ follows the reference $v_{ref}$. As in [Bilalovic et al., 1983; Venkatakrishnan et al., 1985; Carpita et al., 1988], we define, for constant $v_{ref}$,

$$s(t) \equiv (x_1(t) - v_{ref}) + k_0 x_1(t)$$

where $k_0$ is a time constant. Given initial conditions $(x_1^0, x_2^0)$, the zero average dynamics (ZAD) strategy

![Fig. 1. Circuit of a PWM-controlled power converter.](image-url)
is then to choose $d_k$ such that

$$\int_{kT}^{(k+1)T} s(t)dt = 0,$$  \hspace{1cm} (5)

where $(x_1(t), x_2(t))$ in (4) are solutions of (1)–(3) such that $x_k(kT) = x_k^0$. At any sampling period $(kT, (k+1)T)$, the duty cycle $d_k$ is then defined as:

$$d_k = \max(0, \min(1, d_k)),$$

where $d_k$ is the solution of (5).

We can recast the problem in a different form, which will be useful subsequently. Following Angulo et al. [2005c], we set

$$e(t) = x_1(t) - v_{\text{ref}}$$  \hspace{1cm} (6)

and hence from (4) the ZAD function becomes

$$s(t) = e(t) + k_1 \dot{e}(t)$$  \hspace{1cm} (7)

Transforming to coordinates $x = (e, s)$ the dynamics are still governed by (1), (2), (4) and (5) but now (3) becomes

$$A = \begin{pmatrix} -\frac{1}{k_x} & 1 \\ \gamma - 1 & -k_x \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 \\ 1 - v_{\text{ref}}k_s \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 \\ -v_{\text{ref}}k_s \end{pmatrix}$$  \hspace{1cm} (8)

3. Periodic Orbits in Zero Average Dynamics

In this section, we focus on the periodic solutions of the system given by (1)–(3) with the ZAD strategy (5). The usual procedure to find periodic solutions has three steps.

1. The solution $\phi(T, x_0, d_k)$ of (1) in $[0, T]$ is easily shown to be

$$\phi(T; x_0, d_k) = e^{AT}x_0 + (e^{AT} - I)A^{-1}B_1 + (e^{AT} - e^{AT(1 - \frac{d_k}{k})})A^{-1}B_1, \hspace{1cm} (9)$$

2. But the duty cycle $d_k$ is the solution of a nonlinear equation. In fact, we find:

$$\int_{kT}^{(k+1)T} s(t)dt = (1 - k_x \gamma, k_x) \times \int_{kT}^{(k+1)T} \phi(t, x_0, d_k)dt$$

$$- v_{\text{ref}}T = 0$$  \hspace{1cm} (10)

where the integral of the flow in terms of $x_0$ and $d_k$ is given by

$$\int_{kT}^{(k+1)T} \phi(t, x_0, d_k)dt$$

$$= T(1 - 2d_k)A^{-1}B_1$$

$$+ A^{-1}[(e^{AT} - I)(x_0 + A^{-1}B_1)$$

$$+ 2(e^{(1-\frac{d_k}{k})AT} - e^{AT(1 - \frac{d_k}{k})})A^{-1}B_1] \hspace{1cm} (11)$$

3. If we were able to solve Eq. (10) and to obtain $d_k = d_k(x_0)$, we would substitute it in (9) and, solving the equation

$$\phi(T, x_0, d_k(x_0)) = x_0$$

we would find the initial condition $x_0^\ast$ corresponding to the periodic solution of (1) which satisfies the ZAD strategy.

In this paper we change the order of steps 1–3. The crucial observation is the following. The periodic solution $(x_1^\ast(t), x_2^\ast(t))$ obtained from steps 1–3 satisfies:

$$\int_0^T x_1^\ast(t) - v_{\text{ref}} + k_v x_2^\ast(t)dt = 0,$$

but, being $x_2^\ast(t)$ a $T$-periodic function, this equation is equivalent to

$$\int_0^T x_1^\ast(t)dt = v_{\text{ref}}T. \hspace{1cm} (12)$$

On the other hand, integrating the second equation of (1) we can show that:

$$\int_0^T u(t)dt = v_{\text{ref}}T. \hspace{1cm} (13)$$

So, heuristically, it is clear that, no matter what the value of $x_0^\ast$, knowing that it comes from a periodic solution which satisfies the ZAD strategy means that the corresponding (constant) duty cycle $d$ satisfies (12), which gives that $d = v_{\text{ref}}$.

In the following lemma we make these ideas rigorous:

**Lemma 3.1.** Given a linear system

$$\dot{x} = Ax + u$$  \hspace{1cm} (13)

where $A$ is hyperbolic, $u$ is piecewise linear and $L$-periodic, with $\int_0^L u(t)dt = 0$, then there exists a unique $L$-periodic solution $x = x_p(t)$ such that $\int_0^L x_p(t)dt = 0.$
Proof. Define $y = x - U$ where $U(t) = \int_0^t u(\tau) d\tau$. Then $U$ is everywhere continuous, $L$-periodic and differentiable except at points where $u$ is discontinuous. It is straightforward to show that if $x$ is a continuous solution of system (13), $y$ is everywhere differentiable and satisfies $y = Ay + AU'$, a linear system in $y$ with a continuous periodic forced term. Thus, $A$ being a hyperbolic matrix, there exists an unique $L$-periodic solution $y_p$. By periodicity, $\int_0^T y_p dt = 0$ and hence

$$0 = \int_0^T y_p dt = A \int_0^T (y_p + U) dt \quad (14)$$

Then $x_p = y_p + U$ is the unique $L$-periodic solution of (13). Since $A$ is invertible we must have

$$\int_0^T x_p dt = \int_0^T (y_p + U) dt = 0 \quad (15)$$

which completes the proof.

Lemma 3.1 allows us to calculate explicitly the duty cycle, $d_k$, of any periodic solution to the equations in the previous section, without having to find the solution itself.

If we take the control action $u$ in the form given by Eq. (2), with $B_1$ and $B_2$ as in (8), a simple calculation shows that, presuming

$$d_k = d = v_{ref} \quad \forall k, \quad (16)$$

$u$ is $T$-periodic and $\int_0^T u dt = 0$ as well. But Lemma 3.1 implies that $\int_0^T x dt = 0$. So if we work in the transformed variables $(e, s)$, this implies that both $\int_0^T e dt = 0$ and $\int_0^T s dt = 0$. But the latter condition is just the ZAD control strategy. Thus for every $T$-periodic solution of the DC/DC buck converter problem, we have $d = v_{ref}$, independent of the exact form of the solution and, more importantly, independent of the time constant $k_e$.

For centered PWM, Lemma 3.1 gives the result $d = v_{ref}$. In [Angulo et al., 2005c], this was used as an approximation to the duty cycle. In fact, it is an exact result. Since $d \in [0, 1]$, (16) implies that $v_{ref} \in [0, 1]$. Since $x_1 = w/E$ follows the ZAD strategy, this implies that $w \in [0, E]$.

Finally in this section, we derive the exact form of the initial conditions that give a $T$-periodic solution for the DC/DC buck converter with the ZAD control strategy.

**Lemma 3.2.** Given the linear system (1), where the matrix $A$ and the function $u$ are given by (2), (3) and $d_k = d$, the unique $T$-periodic solution has initial conditions

$$x_0^* = -2^{-1}B_1 + 2(I - e^{AT})^{-1} \times (e^{AT} - e^{AT(\tau - \frac{1}{2})})A^{-1}B_1 \quad (17)$$

In addition, when $d = v_{ref}$, this periodic solution satisfies the ZAD strategy.

Proof. The general solution to the equation $\dot{x} = Ax + B$ with initial condition $x(0) = x_0$ and a constant forced vector $B$ is just $x(t) = (-1 + e^{At})A^{-1}B + e^{At}x_0$. Its repeated, consistent, use in the three sections of the control action (2) gives the result (17).

4. Period Doubling

In this section, we find analytic conditions under which the unique $T$-periodic solution undergoes a (classical) period doubling bifurcation. The Poincaré map $P$, over a sampling period $[kT, (k + 1)T]$ is given by

$$P(x_0) = \phi(T, x_0, d_{k}) = e^{AT}x_0 + (e^{AT} - I)A^{-1}B_1 + 2(e^{AT(\tau - \frac{1}{2})})A^{-1}B_1 \quad (18)$$

and the Jacobian of the Poincaré map, $DP(x_0)$, is given by

$$DP(x_0) = e^{AT} + T(e^{AT} - e^{AT(\tau - \frac{1}{2})})A^{-1}B_1 \frac{\partial d_k}{\partial x_0} \quad (19)$$

The term $\partial d_k/\partial x_0$ can be obtained implicitly from the equation for the ZAD strategy, given by

$$\int_{kT}^{(k+1)T} \phi(t, x, d_k) dt = 0 \quad (20)$$

When evaluating $\partial d_k/\partial x_0$ at $x = x_0^*$ given in (17), we use the result that $d_k = d = v_{ref}$.

The period doubling bifurcation curve in the $(v_{ref}, k_e)$ plane is given by

$$m_{pd}(v_{ref}, k_e) \equiv \det(DP(x_0^*) + I) = 0 \quad (21)$$

It is easy to check numerically that the period doubling bifurcation is supercritical. Two points of particular interest, namely the intersections of the
curve with the lines $v_{ref} = 0, 1$. Thus, we define $k^*_+$ and $k^*_-$ such that

\begin{align*}
m_{pd}(1, k^*_+) &= 0 \\
m_{pd}(0, k^*_-) &= 0
\end{align*}

It can also be shown that $(\partial m_{pd}/\partial v_{ref})(1, k^*_+) = 0$, so that the curve $m_{pd}(v_{ref}, k_0)$ intersects the line $v_{ref} = 1$ with vertical slope. Note that for $T = 0.1767, \gamma = 0.35$ we have that $k^*_+ = 2.8478517$ and $k^*_- = 2.8497016$.

5. Nonsmooth Bifurcation

After the period doubling bifurcation occurs ($k_0 > k^*_k$), it is possible that one of these two duty cycles saturates as $k_0$ increases, that is, $d_1$ becomes 1 or $d_2$ becomes 0. When this occurs we have a nonsmooth bifurcation.

This bifurcation curve can be found analytically as follows. If we look for a $(1, d)$ solution of system (1), we have to consider the $2T$-periodic function

\begin{equation}
\begin{cases}
B_1, & 0 \leq t < T + \frac{Td}{2} \\
B_2, & T + \frac{Td}{2} \leq t < 2T - \frac{Td}{2} \\
B_1, & 2T - \frac{Td}{2} \leq t < 2T
\end{cases}
\end{equation}

with $B_1, B_2$ given in (8). We want the solution to satisfy the ZAD strategy in both periods. First, we use Lemma 3.1 with $L = 2T$. Since \( \int_0^{2T} \alpha dt = 0 \), we have from (22) that

\begin{equation}
d = 2v_{ref} - 1
\end{equation}

which replaces (16) in what follows. Then, Lemma 3.1 implies the existence of a $2T$-periodic solution $(e^{**}, s^{**})$ verifying $\int_0^{2T} e^{**} = \int_0^{2T} s^{**} = 0$. Note that $d \in [0, 1]$ only for $v_{ref} \in [1/2, 1]$.

Moreover, a direct calculation shows that $G_{\alpha}(1, k^*_+)$ is linear in $k_0$, and odd in $v_{ref} - 1$, so that it can be written as

\begin{equation}
m_{pd}(v_{ref}, k_0) = (v_{ref} - 1)G_{\alpha}(v_{ref}, k_0)
\end{equation}

where $G_{\alpha}$ is linear in $k_0$ and is even in $(v_{ref} - 1)$.

Going back to the original variables ($x_1, x_2$), we obtain

Lemma 5.1. Given the linear system (1), where matrix $A$ and the $2T$-periodic function $u$ are given by (3) and (22), the unique $2T$-periodic solution has

\begin{align*}
x_{0}^{**} &= -A^{-1}B_1 + 2(I - e^{2AT})^{-1} \\
&\times (e^{AT/2} - e^{AT(1-d/2)})A^{-1}B_1
\end{align*}

In addition, when $d = 2v_{ref} - 1$, this periodic solution satisfies $\int_0^{2T} s^{**}(t) dt = 0$.

We know that $\int_0^{2T} s^{**}(t) dt = 0$ and so we complete the ZAD strategy if

\begin{equation}
m_{pd}(v_{ref}, k_0) \equiv \int_0^{T} s^{**}(t) dt = (1 - k_0)^{v_{ref} - 1} T
\end{equation}

This equation defines the nonsmooth bifurcation curve in $(v_{ref}, k_0)$ parameter space. From (24), (11) and $d = 2v_{ref} - 1$, it is straightforward to show, for $v_{ref} \in [1/2, 1]$, that

\begin{align*}
m_{pd}(v_{ref}, k_0) &= (1 - k_0)T - v_{ref}T \\
&= (1 - k_0) \left\{ -2 \left( \frac{\cos \frac{AT}{2}}{2} \right) \right\} \\
&\times \sinh \frac{AT(v_{ref} - 1)}{2} \\
&\quad \cdot A^{-2}B_1
\end{align*}

(26)

It is clear that $m_{pd}(v_{ref}, k_0)$ is linear in $k_0$ and odd in $v_{ref} - 1$, so that it can be written as

\begin{equation}
m_{pd}(v_{ref}, k_0) = (v_{ref} - 1)G_{\alpha}(v_{ref}, k_0)
\end{equation}

where $G_{\alpha}$ is linear in $k_0$ and is even in $(v_{ref} - 1)$. Moreover, a direct calculation shows that $G_{\alpha}(1, k^*_+)$ is zero, and

\begin{align*}
\sinh \frac{AT}{2} + \left( \frac{\cos \frac{AT}{2}}{2} \right)^2
\end{align*}

\begin{align*}
k^*_+ = \gamma - \frac{\gamma T}{2} + \frac{\alpha T}{2} \\
\frac{\gamma T}{4} \sin \frac{\alpha T}{2}
\end{align*}

and $\alpha = \sqrt{1 - (\gamma/2)^2}$.
Summarizing, both the period doubling bifurcation curve and the nonsmooth bifurcation curve intersect the line $v_{ref} = 1$ at the same point where both curves are vertical in $(v_{ref}, k_s)$ parameter space.

There are analogous results for $v_{ref} \in [0,1/2]$, when $d = 2v_{ref}$. Namely, that $m_\text{cc}(v_{ref}, k_s) = (v_{ref} + 1)\dot{G}_c(v_{ref}, k_s)$ is linear in $k_s$. Moreover $\dot{G}_c(0, k_s^-) = 0$, and $k_s^-$ is given by:

$$k_s^- = \frac{2\sin \alpha T}{\gamma \sin \alpha T + 2\alpha \sinh \gamma T}$$

In this case, both the period doubling bifurcation curve and the nonsmooth bifurcation curve intersect the line $v_{ref} = 0$ also at the same point, but nonvertically.

6. Two Parameter Bifurcation Diagram

In this section, we plot in $(v_{ref}, k_s)$ parameter space, the period doubling bifurcation curve, given by Eq. (21), together with the corner collision curve, Eq. (26) for $v_{ref} \in [1/2,1]$ and its equivalent for $v_{ref} \in [0,1/2]$. The results are shown in Fig. 2. The period doubling curve is given by $AEF$, the corner collision curve in $v_{ref} \in [1/2,1]$ by $AD$ and in $v_{ref} \in [0,1/2]$ by $DF$. Note the narrow horizontal scale. We shall now discuss the six labeled points individually.

Point A

This is $(1, k_s^+, \alpha)$, where $k_s^+$ is given in (28). The two curves are tangent here and intersect the line $v_{ref} = 1$ normally, thus confirming the result of Angulo et al. [2005b].

Point B

For $T = 0.1767$, $\gamma = 0.35$ there is a point $(\tilde{v}_{ref}, \tilde{k}_s)$, with $\tilde{v}_{ref} = 0.5971594$ and $\tilde{k}_s = 2.8481633$ which belongs to the corner collision curve, where the saturated $2T$-periodic orbit experiences a saddle node bifurcation, that is, where $DP^2(x_{s*}^*)$ has an eigenvalue equal to one ($x_{s*}^*$ are the initial conditions of the $2T$-periodic orbit given in Lemma 5.1). This is a codimension-two nonsmooth bifurcation point. The review paper by Kowalczyk et al. (2006) contains a full discussion of the current state of knowledge.
in this area. A line of saddle node bifurcations (not shown) emanates from \( B \) for \( k_s > k_2 \).

**Point C**

The curves \( m_{uv}(v_{ref}, k_u) \) and \( m_{uv}(v_{ref}, k_s) \) cross at \((v_{ref}, k_u)\), with \( v_{ref} = 0.5774674, k_u = 2.8481818 \) when \( T = 0.1767, \gamma = 0.35 \). However there is no exchange of stability here, since the curves relate to different solutions.

**Points D,E**

\( D \) is simply the point where, at \( v_{ref} = 1/2 \), the two separate corner collision curves meet. They do so continuously but not differentiably. In contrast at \( E \), also on \( v_{ref} = 1/2 \), the period doubling curve is smooth.

**Point F**

This is \((0, k^*_s)\), where \( k^*_s \) is given in (29). The two curves are tangent here, but do not intersect the line \( v_{ref} = 0 \) normally.

### 7. One Parameter Bifurcation Diagrams

There are three different types of behavior in Fig. 2, for fixed \( v_{ref} \), as \( k_s \) is varied. In Fig. 3, for \( v_{ref} \in \,(A,B) \), period doubling \((PD)\) occurs before the corner collision \((CC)\) bifurcation. The upper (unstable) branch of the \( 2T \)-periodic solution saturates at the corner collision \((CC)\).

In Fig. 4, for \( v_{ref} \in \,(B,C) \), period doubling \((PD)\) is still before the corner collision \((CC)\) bifurcation, which is then followed by a saddle node \((SN)\) bifurcation. Both (unstable) branches of the \( 2T \)-periodic solution stabilize via a saddle node \((SN)\) bifurcation to produce an unsaturated \(2T\)-periodic solution, with duty cycles \( (1,2v_{ref} - 1) \).

We found two coexisting stable solutions between the corner collision \((CC)\) and the saddle node \((SN)\) bifurcations. For \( v_{ref} = 0.5875 \) and \( k_s = 2.8483 \) the unsaturated \( 2T \)-periodic solution had \( d_1 = 0.75608 \) and \( d_2 = 0.41891 \) (with eigenvalues 0.883372, 0.999998).

Finally in Fig. 5, for \( v_{ref} \in \,(C,F) \), the order of the bifurcations for increasing \( k_s \) is now changed. Therefore there is a range of \( k_s \) in which an unsaturated \( 2T \)-periodic solution is stable when the \( T \)-periodic solution is unstable. For \( v_{ref} = 0.505 \), the corner collision \((CC)\) occurs at \( k_u = 2.8482212 \) and the period doubling \((PD)\) at \( k_s = 2.8483047 \). At \( k_s = 2.84826 \), we found this stable solution with \( d_1 = 0.99981, d_2 = 0.01139 \) (solution eigenvalues are 0.883346 and 0.999956). There is also a range of values, between the period doubling \((PD)\) and saddle node \((SN)\) bifurcations, for which the stable \( T \)-periodic solution occurs at the same time as the unsaturated stable \( 2T \)-periodic solution.
Thus for \( \nu_{\text{ref}} = 0.505 \) and \( k_s = 2.84832 \), that is to the right of the period doubling (PD) curve, we found an unsaturated \( 2T \)-periodic solution with \( d_1 = 0.9679, d_2 = 0.0121 \). This solution was stable up to at least \( k_s = 2.9 \).

8. Conclusions

We have proved a very useful result, Lemma 3.1, for control problems with a strategy based on average quantities. In the case of the DC/DC buck converter with the ZAD strategy, this result allows us to calculate exactly the duty cycle \( d \) of a periodic system response, in the absence of detailed knowledge of the dynamics of the problem. We also calculated the period doubling bifurcation curve of a \( T \)-periodic response in the \((\nu_{\text{ref}}, k_s)\) plane, given exact results for points on the curve in the limiting cases \( \nu_{\text{ref}} = 0, 1 \) and calculated the corner collision bifurcation curve where the \( 2T \)-periodic solution saturates. The period doubling and corner collision curves intersect tangentially at \( \nu_{\text{ref}} = 0, 1 \). We have discovered a codimension two bifurcation point where the corner collision curve collides with a saddle node bifurcation. In practice, owing to the presence of noise, it may be difficult to distinguish all the details in Fig. 2. However, we would expect the overall structure to be maintained. From a design and an implementation point of view, it is clear that the rich dynamics we have shown can entail difficulties in selecting the time constant \( k_s \) or, presuming an orthodox election, in implementing this design.

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