On periodic solutions of 2–periodic Lyness difference equations

Guy Bastien\textsuperscript{1}, \textbf{Víctor Mañosa}\textsuperscript{2} and Marc Rogalski \textsuperscript{3}

\textsuperscript{1}Institut Mathématique de Jussieu, Université Paris 6 and CNRS,

\textsuperscript{2}DMA3-CoDALab, Universitat Politècnica de Catalunya\textsuperscript{*}.

\textsuperscript{3}Laboratoire Paul Painlevé, Université de Lille 1; Université Paris 6 and CNRS,

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We study the \textit{set of periods} of the 2-periodic Lyness’ equations

\begin{equation}
    u_{n+2} = \frac{a_n + u_{n+1}}{u_n},
\end{equation}

where

\begin{equation}
    a_n = \begin{cases} 
    a & \text{for } n = 2\ell + 1, \\
    b & \text{for } n = 2\ell,
\end{cases}
\end{equation}

and being \((u_1, u_2) \in \mathbb{Q}^+; \ell \in \mathbb{N}\) and \(a > 0, b > 0\).

This can be done using the \textit{composition map}:

\begin{equation}
    F_{b,a}(x, y) := (F_b \circ F_a)(x, y) = \left( \frac{a + y}{x}, \frac{a + bx + y}{xy} \right),
\end{equation}

where \(F_a\) and \(F_b\) are the Lyness maps: \(F_\alpha(x, y) = (y, \frac{\alpha + y}{x})\). Indeed:

\[(u_1, u_2) \xrightarrow{F_a} (u_2, u_3) \xrightarrow{F_b} (u_3, u_4) \xrightarrow{F_a} (u_4, u_5) \xrightarrow{F_b} (u_5, u_6) \xrightarrow{F_a} \cdots\]
The map $F_{b,a}$:

- Is a QRT map whose first integral is (Quispel, Roberts, Thompson; 1989):

$$V_{b,a}(x, y) = \frac{(bx + a)(ay + b)(ax + by + ab)}{xy},$$

see also (Janowski, Kulenović, Nurkanović; 2007) and (Feuer, Janowski, Ladas; 1996).

- Has a unique fixed point $(x_c, y_c) \in Q^+$, which is the unique global minimum of $V_{b,a}$ in $Q^+$.

- Setting $h_c := V_{b,a}(x_c, y_c)$, for $h > h_c$ the level sets $\{V_{b,a} = h\} \cap Q^+$ are the closed curves.

$$C_h^+ := \{(bx + a)(ay + b)(ax + by + ab) - hxy = 0\} \cap Q^+ \text{ for } h > h_c.$$

The dynamics of $F_{b,a}$ restricted to $C_h^+$ is conjugate to a rotation with associated rotation number $\theta_{b,a}(h)$. 

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Bastien, Mañosa & Rogalski (Paris 6-UPC)

2-periodic Lyness’ equations

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Theorem A

Consider the family $F_{b,a}$ with $a, b > 0$.

(i) If $(a, b) \neq (1, 1)$, then $\exists p_0(a, b) \in \mathbb{N}$ s.t. for any $p > p_0(a, b)$, $\exists$ at least an oval $C_n^+$ filled by $p$–periodic orbits.

(ii) The set of periods arising in the family \{\(F_{b,a}, \; a > 0, \; b > 0\)\} restricted to $Q^+$ contains all prime periods except 2, 3, 4, 6, 10.

Corollary.

Consider the 2–periodic Lyness’ recurrence for $a, b > 0$ and positive initial conditions $u_1$ and $u_2$.

(i) If $(a, b) \neq (1, 1)$, then $\exists p_0(a, b) \in \mathbb{N}$, s.t. for any $p > p_0(a, b)$ $\exists$ continua of initial conditions giving 2$p$–periodic sequences.

(ii) The set of prime periods arising when $(a, b) \in (0, \infty)^2$ and positive initial conditions are considered contains all the even numbers except 4, 6, 8, 12, 20.

If $a \neq b$, then it does not appear any odd period, except 1.

The value $p_0(a, b)$ is computable for an open and dense set in the parameter space.
To compute the allowed periods, the main issues to take into account are:

- The fact that the rotation number function \( \theta_{b,a}(h) \) is continuous in \([h_c, +\infty)\).
- The fact that generically \( \theta_{b,a}(h_c) \neq \lim_{h \to +\infty} \theta_{b,a}(h) \Rightarrow \exists l(a, b), \) a rotation interval.

**Proposition B.**

\[
\lim_{h \to h^+_c} \theta_{b,a}(h) = \sigma(a, b) := \frac{1}{2\pi} \arccos \left( \frac{1}{2} \left[ -2 + \frac{1}{x_c y_c} \right] \right), \quad \text{and} \quad \lim_{h \to +\infty} \theta_{b,a}(h) = \frac{2}{5}.
\]

**Corollary**

Set \( l(a, b) := \left\langle \sigma(a, b), \frac{2}{5} \right\rangle \).

- If \( \sigma(a, b) \neq \frac{2}{5} \forall \theta \in l(a, b), \) \( \exists \) an oval \( C^+_h \) s.t. \( F_{b,a}(C^+_h) \) is conjugate to a rotation, with a rotation number \( \theta_{b,a}(h) = \theta \).
- In particular, \( \forall \) irreducible \( q/p \in l(a, b), \) \( \exists \) periodic orbits of \( F_{b,a} \) of prime period \( p \).
The periods of the family $F_{b,a}$.

Using the previous results with the family $a = b^2$ we found that:

$$
\bigcup_{b > 0} l(b^2, b) = \left( \frac{1}{3}, \frac{1}{2} \right) \subset \bigcup_{a > 0, b > 0} l(a, b) \subset \bigcup_{a > 0, b > 0} \text{Image} \left( \theta_{b,a}(h_c, +\infty) \right).
$$

**Proposition.**

- For each $\theta$ in $(1/3, 1/2)$ $\exists a, b > 0$ and an oval $C_h^+$, s.t. $F_{b,a}(C_h^+)$ is conjugate to a rotation with rotation number $\theta_{b,a}(h) = \theta$.

- In particular, $\forall$ irreducible $q/p \in (1/3, 1/2)$, $\exists$ $p$-periodic orbits of $F_{b,a}$

We’ll know some periods of $\{F_{b,a}, a, b > 0\}$

$\iff$

We know which are the irreducible fractions in $(1/3, 1/2)$
Lemma (Cima, Gasull, M; 2007)

Given \((c, d)\); Let \(p_1 = 2, p_2 = 3, p_3, \ldots, p_n, \ldots\) be all the prime numbers.

- Let \(p_{m+1}\) be the smallest prime number satisfying that \(p_{m+1} > \max(3/(d - c), 2)\).
- Given any prime number \(p_n, 1 \leq n \leq m\), let \(s_n\) be the smallest natural number such that \(p_n^{s_n} > 4/(d - c)\).
- Set \(p_0 := p_1^{s_1-1}p_2^{s_2-1}\ldots p_m^{s_m-1}\).

Then, for any \(p > p_0\) \(\exists\) an irreducible fraction \(q/p\) s.t. \(q/p \in (c, d)\).

Proof of Theorem A (ii):

- We apply the above result to \((1/3, 1/2)\). \(\forall p \in \mathbb{N}, \text{s.t. } p > p_0\)

\[ p_0 := 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 12252240, \]

\(\exists\) an irreducible fraction \(q/p \in (1/3, 1/2)\).

- A finite checking determines which values of \(p \leq p_0\) s.t. \(q/p \in (1/3, 1/2)\), resulting that there appear irreducible fractions with all the denominators except 2, 3, 4, 6 and 10.

- Proposition C \(\implies\) \(\exists a, b > 0\) s.t. \(\exists\) an oval with rotation number \(\theta_{b,a}(h) = q/p\), thus giving rise to \(p\)–periodic orbits of \(F_{b,a}\) for all allowed \(p\).

- Still it must be proved that 2, 3, 4, 6 and 10 are forbidden, since

\[ I(a, b) \subseteq \text{Image} \left( \theta_{b,a} \left( h_c, +\infty \right) \right) \]
Continuity and asymptotic behavior of $\theta_{b,a}(h)$.

The curves $C_h$, in homogeneous coordinates $[x : y : t] \in \mathbb{C}P^2$, are

$$\tilde{C}_h = \{(bx + at)(ay + bt)(ax + by + abt) - hxyt = 0\}.$$ 

The points $H = [1 : 0 : 0]$, $V = [0 : 1 : 0]$, $D = [b : -a : 0]$ are common to all curves.

**Proposition**

*If $a > 0$ and $b > 0$, and for all $h > h_c$, the curves $\tilde{C}_h$ are elliptic.*
$F_{b,a}$ extends to $\mathbb{C}P^2$ as $\tilde{F}_{b,a}([x : y : t]) = [ayt + y^2 : at^2 + bxt + yt : xy]$.

**Lemma.** Relation between the dynamics of $F_{b,a}$ and the group structure of $\tilde{C}_h$ (*)

For each $h$ s.t. $\tilde{C}_h$ is elliptic,

$$\tilde{F}_{b,a} \mid_{\tilde{C}_h} (P) = P + H$$

Where $+$ is the addition of the group law of $\tilde{C}_h$ taking the infinite point $V$ as the zero element.

Observe that

$$F^n(P) = P + nH,$$

so $\tilde{C}_h$ is full of $p$-periodic orbits $\iff\ pH = V$

i.e. $H$ is a torsion point of $\tilde{C}_h$.

(*) Birational maps preserving elliptic curves can be explained using its group structure (Jogia, Roberts, Vivaldi; 2006).
Instead of looking to a normal form for $F$ we look for a normal form for $\tilde{C}_h$.

$$
\left(\tilde{C}_h, +, \hat{V}\right) \xrightarrow{\sim} \left(\tilde{E}_L, +, \hat{V}\right)
$$

Where $\tilde{E}_L$ is the Weierstrass Normal Form which in the affine plane is:

$$
\mathcal{E}_L = \{ y^2 = 4x^3 - g_2x - g_3 \}
$$

with $g_i := g_i(a, b, h)$.

**WHY?**

1. Because we can parameterize it using the Weierstrass $\wp$ function...
2. ...that gives an integral expression for the rotation number function.

$$
2\Theta(L) = \int_{e_1}^{+\infty} \int_{e_1}^{+\infty} \frac{ds}{\sqrt{4s^3 - g_2s - g_3}}
$$

where $\theta_{b,a}(h) \sim \Theta(L)$

3. The asymptotics of this integral expression can be studied.

This scheme was used in (Bastien, Rogalski; 2004).
The Weierstrass normal form of $C_h$ is

$$E_L = \{ y^2 = 4x^3 - g_2x - g_3 \}$$

where

$$g_2 = \frac{1}{192} \left( L^8 + \sum_{i=4}^{7} p_i(\alpha, \beta)L^i \right) \quad \text{and} \quad g_3 = \frac{1}{13824} \left( -L^{12} + \sum_{i=6}^{11} q_i(\alpha, \beta)L^i \right),$$

being

$$p_7(a, b) = -4(\alpha + \beta + 1),$$
$$p_6(a, b) = 2\left(3(\alpha - \beta)^2 + 2(\alpha + \beta) + 3\right),$$
$$p_5(a, b) = -4(\alpha + \beta - 1)\left(\alpha^2 - 4\beta\alpha + \beta^2 - 1\right),$$
$$p_4(a, b) = (\alpha + \beta - 1)^4.$$

and

$$q_{11}(a, b) = 6(\alpha + \beta + 1),$$
$$q_{10}(a, b) = 3\left(-5\alpha^2 + 2\alpha\beta - 5\beta^2 - 6\alpha - 6\beta - 5\right),$$
$$q_9(a, b) = 4\left(5\alpha^3 - 12\alpha^2\beta - 12\alpha\beta^2 + 5\beta^3 + 3\alpha^2 - 3\alpha\beta + 3\beta^2 + 3\alpha + 3\beta + 5\right),$$
$$q_8(a, b) = 3\left(-5\alpha^4 + 16\alpha^3\beta - 30\alpha^2\beta^2 + 16\alpha^3\beta^3 - 5\beta^4 + 4\alpha^3 \right.$$
$$\left.- 12\alpha^2\beta - 12\alpha\beta^2 + 4\beta^3 + 2\alpha^2 - 8\alpha\beta + 2\beta^2 + 4\alpha + 4\beta - 5\right),$$
$$q_7(a, b) = 6\left(\alpha^2 - 4\alpha\beta + \beta^2 - 1\right)\left(\alpha + \beta - 1\right)^3,$$
$$q_6(a, b) = -(\alpha + \beta - 1)^6.$$

where $\alpha = a/b^2$ and $b/a^2$ and $L \to +\infty \iff h \to +\infty$. 
Since $\tilde{\mathcal{E}}_L \cong \mathbb{T}^2 = \mathbb{C}/\Lambda$, the Weierstrass $\wp$ function relative to a lattice $\Lambda$ gives the parametrization of $\tilde{\mathcal{E}}_L$.

$$\phi : \mathbb{T}^2 = \mathbb{C}/\Lambda \longrightarrow \tilde{\mathcal{E}}_L$$

$$z \longrightarrow \left\{ [\wp(z) : \wp'(z) : 1] \text{ if and } z \notin \Lambda; \ 0 : 1 : 0 = \hat{\mathcal{V}} \text{ if } z \in \Lambda, \right\}$$

Hence $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ since $y^2 = 4x^3 - g_2x - g_3$, and integrating on $[0, u]$:

$$u = \int_{\wp(u)}^{+\infty} \frac{ds}{\sqrt{4s^3 - g_2s - g_3}}$$

- $\hat{G}|_{\mathcal{E}_L}$ is a rotation with $\Theta(L) \in \left[0, \frac{1}{2}\right)$, and $\hat{G}|_{\mathcal{E}_L}(\hat{\mathcal{V}}) = \hat{\mathcal{V}} + \hat{H} = \hat{H}$
- $\hat{H}$ has negative ordinate $\Rightarrow$ is given by $u = 2\omega_1 \Theta(L)$ and its abscissa is $X(L) = \wp(2\omega_1 \Theta(L))$. Hence from $(\ast)$:

$$2\omega_1 \Theta(L) = \int_{X(L)}^{+\infty} \frac{ds}{\sqrt{4s^3 - g_2s - g_3}} \Rightarrow 2\Theta(L) = \int_{e_1}^{+\infty} \frac{ds}{X(L)\sqrt{4s^3 - g_2s - g_3}} \approx \frac{4}{5}.$$
References.

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Other Literature

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THANK YOU!