BEILINSON-FLACH ELEMENTS AND EULER SYSTEMS I: SYNTOMIC REGULATORS AND p-ADIC RANKIN L-SERIES

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Abstract. This article is the first in a series devoted to the Euler system arising from p-adic families of *Beilinson-Flach elements* in the first K-group of the product of two modular curves. It relates the image of these elements under the p-adic syntomic regulator (as described by Besser [Bes3]) to the special values at the near-central point of Hida's p-adic Rankin L-function attached to two Hida families of cusp forms.

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1. Introduction

This article is the first in a series devoted to the Euler system of *Beilinson-Flach elements* in the motivic cohomology of a product of two modular curves. Its main result (see Theorem 4.2 and Corollary 4.4 of §4.2) is a p-adic analogue of the formula of Beilinson [Bei, Ch. 2, §6] expressing special values of Rankin L-series in terms of complex regulators. Beilinson's theorem (cf. §4.1 for an explicit version) relates:

(1) the Rankin L-series $L(f \otimes g, s)$ attached to the convolution of weight 2 newforms f and g on $\Gamma_1(N)$, evaluated at the *near-central point* $s = 2$;
(2) the image under the complex regulator of certain explicit elements in the motivic cohomology group $H^3_{\mathcal{M}}(X_1(N)^2, \mathbb{Q}(2))$, or, equivalently, in the higher Chow group $\mathrm{CH}^2(X_1(N)^2, 1) \otimes \mathbb{Q}$. These elements, whose definition is recalled in Section 3.1, are constructed from modular units and are referred to in the sequel as *Beilinson-Flach elements.*

In the p-adic setting, the complex L-series $L(f \otimes g, s)$ is replaced by Hida's p-adic Rankin L-series attached to two ordinary families of modular forms interpolating f and g, whose definition is briefly recalled in Section 2.2. The role of the complex regulator is played by the p-adic syntomic regulator on K_1 of a surface. Besser's description of it in terms of Coleman integration [Bes3], which is summarised in §3.3, is a key ingredient in the proof of Theorem 4.2.

Our approach also relies crucially on techniques developed in [DR] for relating p-adic Abel-Jacobi images of diagonal cycles to values of the Garrett-Rankin triple product p-adic L-function attached to a triple $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ of Hida families of cusp forms. Corollary 4.4 deals with the setting where the cuspidal family $\mathbf{h}$ in the triple $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ is replaced by a Hida family of Eisenstein series. The reader will also note the close parallel between Theorem 4.2 and the main result of [BD], in which the p-adic regulators of certain elements in $K_2(X_1(N))$ are related to the value at $s = 2$ of the Mazur-Swinnerton-Dyer p-adic L-functions attached to weight two cusp forms. The results of the present article are in fact intermediate between those of [DR] and [BD], the latter treating the case where *both* $\mathbf{g}$ and $\mathbf{h}$ are replaced by Hida families of Eisenstein series—a setting in which the resulting p-adic Rankin L-function factors as a product of two Mazur-Kitagawa L-functions attached to $\mathbf{f}$.

We also remark that a function field analogue of Beilinson's Theorem involving Drinfeld modular curves is described in [Sre2], based on a description of non-archimedean regulators given in [Sre1]. See also the related work of Ambrus Pàl in the setting of the K_2 of Mumford curves [Pa].

Let us conclude this introduction by briefly discussing some eventual arithmetical applications of the main result of this paper.

I. The Euler system of Beilinson-Flach elements. The image of Beilinson-Flach elements under the p-adic étale regulator map gives rise to classes in the global cohomology group $H^1(\mathbb{Q}, V_f \otimes V_g(2))$, where V_f and V_g are the p-adic Galois representations attached to f and g, respectively. The work in preparation [BDR] explores the theme of the p-adic variation of the Beilinson-Flach classes attached to Hida families of cusp forms $\mathbf{f}$ and $\mathbf{g}$. In particular, when $\mathbf{g}$ specialises in weight one to a classical cusp form attached to an odd irreducible Artin representation ρ, and $\mathbf{f}$ specialises in weight two to the cusp form associated with an elliptic curve E over $\mathbb{Q}$, we expect the associated cohomology class to yield new cases of the Birch and Swinnerton-Dyer conjecture for the complex L-series $L(E, \rho, s)$, proving in particular that ρ does not occur in the representation $E(\bar{\mathbb{Q}}) \otimes \mathbb{C}$ when $L(E, \rho, 1) \neq 0$.

The idea of using Beilinson elements in Euler system arguments occurs much earlier in the work of Flach [Fl], who used them to construct classes in $H^1(\mathbb{Q}, \mathrm{Sym}^2(E)(2))$ which are cristalline at p but ramified at a single prime $\ell \neq p$. Applying Kolyvagin's method to these classes leads to the finiteness of the Shafarevich-Tate group of $\mathrm{Sym}^2(E)(2)$ and an explicit annihilator of this group related to the special value $L(\mathrm{Sym}^2(E), 2)$, which is critical in the sense of Deligne, unlike the special values $L(f \otimes g, 2)$ when f and g are distinct normalised newforms.

II. Hida's L-function for the symmetric square of a modular form. Theorem 4.2, specialised to the case $f = g$, is exploited by S. Dasgupta [Das] to study the Hida L-function $L(f \otimes f, s)$ and express it as the product of the Coates-Schmidt p-adic L-function attached to $\mathrm{Sym}^2(f)$ and a Kubota-Leopoldt p-adic L-function. This factorisation, which can be viewed as another manifestation of the Artin formalism for p-adic L-series, is analogous to a formula of Gross [Gross] expressing the restriction to the cyclotomic line of the Katz two-variable p-adic L-function attached to an imaginary quadratic field as a product of two Kubota-Leopoldt L-functions. The Beilinson-Flach elements play the same role in Dasgupta's proof as elliptic units in the work of Gross.

III. p-adic L-functions and Euler systems over $\mathbb{Z}_p^2$-extensions. The paper in preparation [LLZ] of A. Lei, D. Loeffler and S.L. Zerbes builds on the methods of this paper, in the setting where g varies over a collection of theta series attached to Hecke characters of an imaginary quadratic field K, to construct an Euler system for V_f over the various layers of the *two variable* $\mathbb{Z}_p$-extension K_∞ of K, thus supplying the global input for their extension [LZ] of Perrin-Riou's machinery in which the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ is replaced by K_∞.

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2. Rankin L-series

Let f and g be normalised newforms of weights k, ℓ, levels N_f, N_g, and nebentypus characters χ_f, χ_g respectively. The p-adic representations V_f and V_g are part of a compatible system of representations which we continue to denote by the same symbol. Let

$$L(V_f \otimes V_g, s) = \prod_p \det((1 - \sigma_p p^{-s})|(V_f \otimes V_g)^{I_p})^{-1}$$

be the motivic L-function attached to the tensor product $V_f \otimes V_g$, where I_p denotes the inertia subgroup of a decomposition group at p, and σ_p a corresponding geometric Frobenius element.

The goal of this first chapter is to briefly recall the basic analytic properties of this L-series, describe Hida's construction of a p-adic avatar, and— in the special case where f and g are both of weight two—present parallel formulae for their special values at the near-central point $s = 2$, which is *not* critical in the sense of Deligne.

2.1. Complex L-series.

Set $N := \mathrm{lcm}(N_f, N_g)$ and replace χ_f and χ_g by their counterparts of modulus N sending any prime $r|N$ to 0. It is also convenient to replace f, as well as g, by a normalised eigenform of level N which

(1) has the same eigenvalues for the good Hecke operators T_r with $\gcd(r, N) = 1$;
(2) is also an eigenvector for the Hecke operators U_r attached to the primes r dividing N.

This substitution having been made, let

$$f(z) = \sum_{n=1}^{\infty} a_n(f) e^{2\pi i n z}, \qquad g(z) = \sum_{n=1}^{\infty} a_n(g) e^{2\pi i n z} \tag{1}$$

be the Fourier expansions of f and g, let K_f and $K_g \subset \bar{\mathbb{Q}}$ denote the subfields generated by the coefficients $a_n(f)$ and $a_n(g)$ respectively, and let K_{fg} denote the compositum of the two fields. The Hecke polynomials attached to f can be factored as

$$x^2 - a_p(f)x + \chi_f(p)p^{k-1} = (x - \alpha_p(f))(x - \beta_p(f)),$$

where $(\alpha_p(f), \beta_p(f)) = (a_p(f), 0)$ when $p|N$. Similar notations are adopted for g. The Rankin L-function attached to the pair (f, g) is defined by the formula

$$\begin{aligned} L(f \otimes g, s) \ &:= \ \prod_p L_{(p)}(f \otimes g, s), \qquad \text{where} \\ L_{(p)}(f \otimes g, s) \ &:= \ (1 - \alpha_p(f)\alpha_p(g)p^{-s})^{-1}(1 - \alpha_p(f)\beta_p(g)p^{-s})^{-1} \\ &\quad \times (1 - \beta_p(f)\alpha_p(g)p^{-s})^{-1}(1 - \beta_p(f)\beta_p(g)p^{-s})^{-1}. \end{aligned}$$

The Euler factors at p defining $L(V_f \otimes V_g, s)$ and $L(f \otimes g, s)$ agree for all $p \nmid N$, and hence the special values of $L(V_f \otimes V_g, s)$ and $L(f \otimes g, s)$ at integer points differ by elementary quantities in $K_{fg}^\times$. It will be more convenient, for the sequel, to focus attention on $L(f \otimes g, s)$. Assume without loss of generality that the forms f and g have been ordered in such a way that $k \geq \ell$.

2.1.1. *Rankin's method.* We begin by recalling the general formula for $L(f\otimes g,s)$ coming out of Rankin's method, involving the *non-holomorphic Eisenstein series*

$$\widetilde{E}_{k-\ell,\chi}(z,s) = {\sum_{(m,n)\in N\mathbb{Z}\times\mathbb{Z}}}' \frac{\chi^{-1}(n)}{(mz+n)^{k-\ell}}\cdot\frac{y^s}{|mz+n|^{2s}} \tag{2}$$

of weight $k-\ell$, level N and character

$$\chi := \chi_f^{-1}\chi_g^{-1},$$

where the superscript $'$ in (2) indicates that the sum is taken over the non-zero lattice vectors $(m,n)\in N\mathbb{Z}\times\mathbb{Z}$. For fixed complex s with $\mathrm{Re}(s)\gg 0$, the product $\widetilde{E}_{k-\ell,\chi}(z,s)\times g(z)$ is a real-analytic $\mathbb{C}$-valued function on the Poincaré upper half-plane $\mathcal{H}$ which transforms like a modular form of weight k, level N and character χ_f^{-1} and is of rapid decay at infinity. The space of such functions, denoted $S_k^{\mathrm{ra}}(N,\chi_f^{-1})$, is equipped with the Petersson scalar product

$$\langle\ ,\ \rangle_{k,N} : S_k^{\mathrm{ra}}(N,\chi_f^{-1})\times S_k^{\mathrm{ra}}(N,\chi_f^{-1}) \longrightarrow \mathbb{C}$$

given by the formula

$$\langle f_1,f_2\rangle_{k,N} := \int_{\Gamma_0(N)\backslash\mathcal{H}} y^k\overline{f_1(z)}f_2(z)\frac{dxdy}{y^2}, \tag{3}$$

which is hermitian linear in the first argument and $\mathbb{C}$-linear in the second. Let $f^*\in S_k(N,\chi_f^{-1})$ denote the modular form obtained from f by applying complex conjugation to its Fourier coefficients.

Proposition 2.1 (Shimura). *For all $s\in\mathbb{C}$ with $\Re(s)\gg 0$,*

$$L(f\otimes g,s) = \frac{1}{2}\frac{(4\pi)^s}{\Gamma(s)}\left\langle f^*(z),\widetilde{E}_{k-\ell,\chi}(z,s-k+1)\cdot g(z)\right\rangle_{k,N}. \tag{4}$$

This well-known formula for the Rankin L-series is taken from equation (14) of [BD].

2.1.2. *Critical values.* Assume here and in §2.1.3 that $\ell<k$. The functional equation for $L(f\otimes g,s)$ arising from Proposition 2.1 reveals that the integer j is critical for $L(f\otimes g,s)$ if any only if it lies in the closed interval $[\ell,k-1]$. We now describe a further closed formula for the value at an integer j belonging to the "right half critical segment" $[\frac{\ell+k-1}{2},k-1]$, which will be useful in deriving the algebraicity (up to periods) of $L(f\otimes g,j)$ predicted by the Deligne conjectures, and ultimately in constructing Hida's p-adic Rankin L-function by interpolating these quantities p-adically.

Having fixed an integer $j\in[\frac{\ell+k-1}{2},k-1]$, let $t\geq 0$ and $m\geq 1$ be given by

$$t := k-1-j, \qquad m := k-\ell-2t.$$

If $m\leq 2$, let us assume also that χ is nontrivial. Then the series

$$E_{m,\chi}(z) = 2^{-1}L(\chi,1-m)+\sum_{n=1}^{\infty}\sigma_{m-1,\chi}(n)q^n, \qquad \sigma_{m-1,\chi}(n)=\sum_{d|n}\chi(d)d^{m-1} \tag{5}$$

is the q-expansion of a *holomorphic* Eisenstein series of weight m and character χ.

The Shimura-Maass derivative operator

$$\delta_m := \frac{1}{2\pi i}\left(\frac{d}{dz}+\frac{im}{2y}\right)$$

transforms modular forms of weight m into (real analytic) modular forms of weight $m+2$ which are *nearly holomorphic* in the sense of [Sh2], and its t-fold iterate $\delta_m^t := \delta_{m+2t-2}\cdots\delta_{m+2}\delta_m$

maps the space $M_m(N,\chi)$ to the space $M^{\mathrm{nh}}_{m+2t}(N,\chi)$ of nearly holomorphic modular forms of weight $m+2t$. Let

$$C(k,\ell,j) := \frac{(-1)^t 2^{k-1}(2\pi)^{k+m-1}\iota_\chi (iN)^{-m}\tau(\chi^{-1})}{(m+t-1)!(j-1)!}$$

be the elementary constant (in which $\iota_\chi = 1$ when χ is primitive) appearing in equation (18) of [BD]. The following formula for $L(f\otimes g,j)$, is obtained by setting $c=j$ in equation (18) of loc. cit. (See also Theorem 2 of [Sh1].)

Proposition 2.2. *The special value $L(f\otimes g,j)$ is given by the formula*

$$L(f\otimes g,j) = C(k,\ell,j)\left\langle f^*(z), \delta_m^t E_{m,\chi}(z)\times g(z)\right\rangle_{k,N}. \tag{6}$$

2.1.3. *Algebraicity and Deligne's conjecture.* Let $S_k^{\mathrm{nh}}(N,\chi_f^{-1};K_{fg}) \subset S_k^{\mathrm{nh}}(N,\chi_f^{-1})$ denote the space of nearly-holomorphic cusp forms which are *defined over K_{fg}* in the sense of Shimura (cf. Section 2.4 of [DR]). The cusp form

$$\Xi(f,g,j) := \delta_m^t E_{m,\chi}\times g \in S_k^{\mathrm{nh}}(N,\chi_f^{-1}) \tag{7}$$

which appears in Proposition 2.2 belongs to the K_{fg}-rational structure $S_k^{\mathrm{nh}}(N,\chi_f^{-1};K_{fg})$. Hence, its image

$$\Xi(f,g,j)^{\mathrm{hol}} := \Pi_N^{\mathrm{hol}}(\Xi(f,g,j)) \tag{8}$$

under the holomorphic projection Π_N^{hol} of loc. cit. belongs to the space $S_k(N,\chi_f^{-1};K_{fg})$ of holomorphic cusp forms with Fourier coefficients in K_{fg}. In particular, the ratio

$$L^{\mathrm{alg}}(f\otimes g,j) := C(f,g,j)^{-1}\frac{L(f\otimes g,j)}{\langle f^*,f^*\rangle_{k,N}} = \frac{\langle f^*,\Xi(f,g,j)\ \rangle_{k,N}}{\langle f^*,f^*\rangle_{k,N}} = \frac{\left\langle f^*,\Xi(f,g,j)^{\mathrm{hol}}\right\rangle_{k,N}}{\langle f^*,f^*\rangle_{k,N}}, \tag{9}$$

belongs to K_{fg}. This algebraicity result is consistent with Deligne's conjecture which predicts that the period $C(f,g,j)\langle f^*,f^*\rangle_{k,N}$ is the 'transcendental part" of the special value $L(f\otimes g,j)$. The associated "algebraic part" appearing in (9) will later be interpolated p-adically to obtain Hida's p-adic Rankin L-function attached to f and g.

In order to do this, it will be convenient to give a more geometric description of the quantity $L^{\mathrm{alg}}(f\otimes g,j)$ appearing in (9), in terms of the Poincaré duality on the de Rham cohomology of the modular curve $X_1(N)$ with values in appropriate sheaves with connection, as described in [DR, § 2.3]. To lighten the notations, denote by Y and by X the open modular curve $Y_1(N)$ and the complete modular curve $X_1(N)$ respectively, classifying (generalised) elliptic curves equipped with an embedding of the finite flat group scheme μ_N of N-th roots of unity.

Let K be any field containing K_{fg}. Denote by $\mathcal{E}\longrightarrow Y$ the universal elliptic curve over Y, and by ω the sheaf of relative differentials on $\mathcal{E}$ over Y, extended to X as in [BDP, § 1.1]. Recall the Kodaira-Spencer isomorphism $\omega^2 \simeq \Omega^1_X(\log\mathrm{cusps})$, where $\Omega^1_X(\log\mathrm{cusps})$ is the sheaf of regular differentials on Y with log poles at the cusps.

A modular form ϕ on $\Gamma_1(N)$ of weight $k=r+2$ with Fourier coefficients in K corresponds to a global section of the sheaf $\omega^{r+2} = \omega^r\otimes\Omega^1_X(\log\mathrm{cusps})$ over the base-change X_K of X to K. The sheaf ω^r can be viewed as a subsheaf of $\mathcal{L}_r := \mathrm{Sym}^r\,\mathcal{L}$, where

$$\mathcal{L} := R^1\pi_*(\mathcal{E}\longrightarrow Y)$$

is the relative de Rham cohomology sheaf on Y, extended to X as in loc. cit., equipped with the filtration

$$0\longrightarrow\omega\longrightarrow\mathcal{L}\longrightarrow\omega^{-1}\longrightarrow 0 \tag{10}$$

arising from the Hodge filtration on the fibers. The sheaf $\mathcal{L}_r$ is a coherent sheaf over X of rank $r+1$, endowed with the Gauss-Manin connection

$$\nabla : \mathcal{L}_r \longrightarrow \mathcal{L}_r\otimes\Omega^1_X(\log\mathrm{cusps}).$$

Let $H^1_{\mathrm{dR}}(X_K,\mathcal{L}_r,\nabla)$ be the de Rham cohomology of $\mathcal{L}_r$. It is equipped with the perfect Poincaré pairing

$$\langle\ ,\ \rangle_{k,X} : H^1_{\mathrm{dR}}(X_K,\mathcal{L}_r,\nabla)\times H^1_{\mathrm{dR}}(X_K,\mathcal{L}_r,\nabla)\longrightarrow K, \tag{11}$$

which is compatible with the exact sequence

$$0\longrightarrow H^0(X_K,\omega^r\otimes\Omega^1_X)\longrightarrow H^1_{\mathrm{dR}}(X_K,\mathcal{L}_r,\nabla)\longrightarrow H^1(X_K,\omega^{-r})\longrightarrow 0, \tag{12}$$

in the sense that $H^0(X_K,\omega^r\otimes\Omega^1_X)$ is an isotropic subspace. (Cf. Sections 2 and 3 of [Col], for a more detailed account.) In particular, Poincaré duality induces a perfect pairing

$$\langle\ ,\ \rangle_{k,X} : H^1(X_K,\omega^{-r})\times H^0(X_K,\omega^r\otimes\Omega^1_X)\longrightarrow K, \tag{13}$$

which is denoted by the same symbol by a slight abuse of notation.

Set $\omega_f = f(z)dz$ and $\overline{\omega}_f = \bar{f}^*(z)d\bar{z}$. The antiholomorphic differential η_f^{ah} defined by

$$\eta_f^{\mathrm{ah}} := \frac{\overline{\omega}_f}{\langle\overline{\omega}_f,\omega_f\rangle_{k,X}}\ . \tag{14}$$

gives rise to a class in $H^1_{\mathrm{dR}}(X_{\mathbb{C}},\mathcal{L}_r,\nabla)$, whose image η_f in $H^1(X_{\mathbb{C}},\omega^{-r})$ belongs to $H^1(X_K,\omega^{-r})$ (cf. Corollary 2.13 of [DR]). The following expression for the algebraic part $L^{\mathrm{alg}}(f\otimes g,j)$ in terms of the class η_f follows directly from (9) in light of the discussion above:

Proposition 2.3. *The algebraic part $L^{\mathrm{alg}}(f\otimes g,j)$ is equal to*

$$L^{\mathrm{alg}}(f\otimes g,j) = \left\langle \eta_f,\Xi(f,g,j)^{\mathrm{hol}}\right\rangle_{k,X}. \tag{15}$$

2.1.4. *The value at the near central point.* Consider now the case where $k=\ell=2$ and $\chi_f\neq\chi_g^{-1}$, so that the character $\chi=\chi_f^{-1}\chi_g^{-1}$ is not the trivial one. The functional equation for $L(f\otimes g,s)$ relates $L(f\otimes g,s)$ to $L(f^*\otimes g^*,3-s)$ and this L-series has no critical points in the sense of Deligne. Proposition 2.5 below describes its value at the near-central point $s=2$ in terms of logarithms of *modular units.*

Enlarge K so that it contains the field which is cut out by all the Dirichlet characters of modulus N, and let F be the field generated over K by the values of these characters. Let $\mathrm{Eis}_\ell(\Gamma_1(N);F)$ denote the subspace of $M_\ell(\Gamma_1(N);F)$ spanned by the weight ℓ Eisenstein series with coefficients in F. The logarithmic derivative gives a surjective homomorphism

$$\mathcal{O}(Y_K)^\times\otimes F \xrightarrow{\ \mathrm{dlog}\ } \mathrm{Eis}_2(\Gamma_1(N);F), \tag{16}$$

whose kernel is the subspace $K^\times\otimes F$ spanned by the nonzero constant functions.

Definition 2.4. Let u_χ be the modular unit satisfying

$$\mathrm{dlog}(u_\chi) = E_{2,\chi} \tag{17}$$

whose value at ∞ is 1 in the sense of [Br, §5].

Proposition 2.5. *Given weight two eigenforms f and g as above,*

$$L(f\otimes g,2) = 16\pi^3N^{-2}\tau(\chi^{-1})\,\langle f^*(z),\log|u_\chi(z)|\cdot g(z)\rangle_{2,N}\,. \tag{18}$$

Proof. By Proposition 2.1,

$$L(f\otimes g,2) = \frac{1}{2}(4\pi)^2\left\langle f^*(z),\widetilde{E}_{0,\chi}(z,1)\cdot g(z)\right\rangle_{2,N}. \tag{19}$$

A direct calculation (cf. equation (26) of [BD]) shows that

$$\frac{1}{2\pi i}\frac{d}{dz}\widetilde{E}_{0,\chi}(z,1) = -\frac{1}{4\pi}\widetilde{E}_{2,\chi}(z) = 2\pi N^{-2}\tau(\chi^{-1})E_{2,\chi}(z). \tag{20}$$

Having normalized u_χ as in Definition 2.4, one obtains the equality

$$\widetilde{E}_{0,\chi}(z,1) = 2\pi N^{-2}\tau(\chi^{-1})\log|u_\chi(z)|, \tag{21}$$

which is compatible with (17). Combining (19) with (21) completes the proof of the proposition. □

2.2. **p-adic L-series.** Let $p \geq 3$ be a prime, and fix an embedding of K into $\mathbb{C}_p$. This section recalls the definition of the Rankin p-adic L-function associated by Hida [Hi] to the convolution of two Hida families of cusp forms. For the sake of brevity, we proceed here—just as in [BD]— by specialising the approach and notations of [DR], which constructs the p-adic L-function associated to a triple product of three Hida families $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ of cusp forms. The setting considered here consists, essentially, in letting $\mathbf{h}$ be a Hida family of Eisenstein series.

2.2.1. *Ordinary projections.* Let f, g be eigenforms of level N, weights $k > \ell$ and nebentypus χ_f, χ_g as in (1). Let also $j \in [\frac{\ell+k-1}{2}, k-1]$ be an integer and set $t = k-1-j \geq 0$ and $m = k-\ell-2t \geq 1$ as in §2.1.2. The following ordinariness assumption is important for the constructions described in this section.

Assumption 2.6. The cuspidal eigenforms f and g are ordinary at p, and $p \nmid N$.

Under this assumption, the f-isotypic part of the exact sequence (12) with $K = \mathbb{C}_p$ admits a canonical *unit root* splitting, arising from the action of Frobenius on de Rham cohomology. Let η_f^{ur} be the lift of η_f to the unit root subspace $H^1_{\mathrm{dR}}(X_{\mathbb{C}_p}, \mathcal{L}_r, \nabla)^{f,\mathrm{ur}}$. The right-hand side of (15) is then equal to

$$\left\langle \eta_f, \Xi(f,g,j)^{\mathrm{hol}} \right\rangle_{k,X} = \left\langle \eta_f^{\mathrm{ur}}, \Xi(f,g,j)^{\mathrm{hol}} \right\rangle_{k,X}. \tag{22}$$

Now let e_{ord} be Hida's ordinary projector to $H^1_{\mathrm{dR}}(Y_K, \mathcal{L}_r, \nabla)^{\mathrm{ord}}$. By Proposition 2.11 of [DR], the right-hand side of (22) can be re-written, after viewing $\Xi(f,g,j)^{\mathrm{hol}}$ as an overconvergent p-adic modular form and setting $\Xi(f,g,j)^{\mathrm{ord}} := e_{\mathrm{ord}}\Xi(f,g,j)^{\mathrm{hol}}$, as

$$\left\langle \eta_f^{\mathrm{ur}}, \Xi(f,g,j)^{\mathrm{hol}} \right\rangle_{k,X} = \left\langle \eta_f^{\mathrm{ur}}, \Xi(f,g,j)^{\mathrm{ord}} \right\rangle_{k,X}. \tag{23}$$

By Proposition 2.8 of [DR],

$$\Xi(f,g,j)^{\mathrm{ord}} = e_{\mathrm{ord}}(d^t E_{m,\chi} \cdot g), \tag{24}$$

where $d = q\frac{d}{dq}$ is Serre's derivative operator on p-adic modular forms.

Given a p-adic modular form $\phi = \sum c_n q^n$, let $\phi^{[p]} := \sum_{p\nmid n} c_n q^n$ denote its "p-depletion", and set

$$\Xi(f,g,j)^{\mathrm{ord},p} := e_{\mathrm{ord}}(d^t E^{[p]}_{m,\chi} \cdot g). \tag{25}$$

Proposition 2.7. *Let e_{f^*} be the projector to the f^*-isotypic subspace of $H^1_{\mathrm{dR}}(Y_K, \mathcal{L}_{k-2}, \nabla)$. Then*

$$e_{f^*}\Xi(f,g,j)^{\mathrm{ord},p} = \frac{\mathcal{E}(f,g,j)}{\mathcal{E}(f)} \cdot e_{f^*}\Xi(f,g,j)^{\mathrm{ord}},$$

where

$$\begin{aligned} \mathcal{E}(f,g,j) &= (1-\beta_p(f)\alpha_p(g)p^{t-k+1})(1-\beta_p(f)\beta_p(g)p^{t-k+1}) \\ &\quad \times(1-\beta_p(f)\alpha_p(g)\chi(p)p^{t-k+m})(1-\beta_p(f)\beta_p(g)\chi(p)p^{t-k+m}), \\ \mathcal{E}(f) &= 1-\beta_p(f)^2\chi_f^{-1}(p)p^{-k}. \end{aligned}$$

Proof. This follows from Corollary 4.17 of [DR], in light of Proposition 2.8 of loc. cit. □

2.2.2. *Hida's p-adic L-series.* Let $\mathbf{f}$ and $\mathbf{g}$ be Hida families of ordinary p-adic modular forms of tame level N, indexed by weight variables k and ℓ in suitable neighborhoods $U_{\mathbf{f}}$ and $U_{\mathbf{g}}$ of $\mathbb{Z}/(p-1)\mathbb{Z}\times\mathbb{Z}_p$, contained in a single residue class modulo $p-1$. (These families may be obtained, as shall be the case considered in §4.2, by deforming two given ordinary classical eigenforms f and g of possibly equal weights.) Assume likewise that the parameter $j = k-1-t$ belongs to a single residue class modulo $p-1$, so that the same holds true for the weight $m = k-\ell-2t$ of the Eisenstein series $E_{m,\chi}$.

For $k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}$ and $\ell \in U_{\mathbf{g}} \cap \mathbb{Z}^{\geq 2}$, let

$$f_k \in S_k(N,\chi_f), \qquad g_\ell \in S_\ell(N,\chi_g)$$

be the classical cusp forms whose p-stabilisations are the weight k and ℓ specialisations of $\mathbf{f}$ and $\mathbf{g}$ respectively. (We denote by χ_f, resp. χ_g the common character of the modular forms f_k, resp. g_k.)

The collection of p-adic modular forms $\Xi(f_k, g_\ell, j)^{\mathrm{ord},p}$ (defined as in equation (25)) indexed by

$$\{(k,\ell,j), \qquad k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}, \quad \ell \in U_{\mathbf{g}} \cap \mathbb{Z}^{\geq 2}, \quad \frac{\ell+k-1}{2} \leq j \leq k-1\} \tag{26}$$

has Fourier coefficients which extend analytically to $U_{\mathbf{f}} \times U_{\mathbf{g}} \times \mathbb{Z}_p$, as functions in k, ℓ and j. Hence, they can be viewed as a (three-variable) Λ-adic family of modular forms of level N in the sense of [DR, §2.7].

Set

$$\mathcal{E}^*(f_k) := 1 - \beta_p(f_k)^2 \chi_f^{-1}(p) p^{1-k}.$$

Proposition 4.10 of loc. cit. shows that the expression

$$L_p(\mathbf{f},\mathbf{g})(k,\ell,j) := \frac{1}{\mathcal{E}^*(f_k)} \left\langle \eta^{\mathrm{ur}}_{f_k}, \Xi(f_k, g_\ell, j)^{\mathrm{ord},p} \right\rangle_{k,X}, \tag{27}$$

defined on the triples (k,ℓ,j) in the set in (26) extends to an analytic function $L_p(\mathbf{f},\mathbf{g})$ on $U_{\mathbf{f}} \times U_{\mathbf{g}} \times \mathbb{Z}_p$, which we refer to as the Hida *p-adic Rankin L-function* attached to $\mathbf{f}$ and $\mathbf{g}$. This appelation is justified by noting that, for all triples (k,ℓ,j) in the range of "classical interpolation", i.e, belonging to (26), the function $L_p(\mathbf{f},\mathbf{g})(k,\ell,j)$ satisfies the interpolation property

$$L_p(\mathbf{f},\mathbf{g})(k,\ell,j) = \frac{\mathcal{E}(f_k, g_\ell, j)}{\mathcal{E}^*(f_k)\mathcal{E}(f_k)} L^{\mathrm{alg}}(f_k \otimes g_\ell, j).$$

This follows from a direct calculation combining (27), Proposition 2.7, (23), (22) and (15).

Note that the point $(2,2,2)$ lies outside the region of classical interpolation for this function. (In fact, there are no critical values for the pair of weights $(2,2)$.) Corollary 4.4 of Section 4.2 relates the value of $L_p(\mathbf{f},\mathbf{g})$ at $(2,2,2)$ to the p-adic regulator attached in Section 3.3 to the triple of modular forms $(f = f_2, g = g_2, E_{2,\chi})$.

Generalising our setting somewhat, we do not assume now that $g \in S_2(N,\chi_g)$ is ordinary, so that g may not necessarily be viewed as the weight 2 specialisation of a Hida family. In this case, the above construction still allows us to define a two-variable p-adic L-function $L_p(\mathbf{f},g)(k,j)$ on $U_{\mathbf{f}} \times \mathbb{Z}_p$, by the equation

$$L_p(\mathbf{f},g)(k,j) := \frac{1}{\mathcal{E}^*(f_k)} \left\langle \eta^{\mathrm{ur}}_{f_k}, \Xi(f_k, g, j)^{\mathrm{ord},p} \right\rangle_{k,X}, \tag{28}$$

for $k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}$ and $(k+1)/2 \leq j \leq k-1$. Theorem 4.2 relates $L_p(\mathbf{f},g)(2,2)$ to the p-adic regulator attached to $(f,g,E_{2,\chi})$.

3. Beilinson-Flach elements

3.1. **Definition and basic properties.** Let S be a quasi-projective variety over a field K, and $K_j(S)$ denote Quillen's algebraic K-groups of S. The motivic cohomology groups $H^i_{\mathcal{M}}(S,\mathbb{Q}(n)) = K^{(n)}_{2n-i}(S)$ of S were defined by Beilinson [Bei, §2] as the n-th graded piece of the Adams filtration on $K_{2n-i}(S)\otimes\mathbb{Q}$. In parallel with Beilinson's motivic cohomology groups, Bloch [Bl] introduced the higher Chow groups $\mathrm{CH}^i(S,n)$ of S.

In this note we shall focus on the smooth projective surface $S := X\times X$, where X is the modular curve over the field K of §2.1.4. For $i=3$ and $n=2$, $H^3_{\mathcal{M}}(S,\mathbb{Q}(2)) = K^{(2)}_1(S)$ is identified with $\mathrm{CH}^2(S,1)\otimes\mathbb{Q}$. The higher Chow group $\mathrm{CH}^2(S,1)$ may be explicitly described (cf. also [Sc]) as the first homology of the *Gersten complex*

$$K_2(K(S)) \xrightarrow{\ \partial\ } \bigoplus_{Z\subset S} K(Z)^\times \xrightarrow{\ \mathrm{div}\ } \bigoplus_{P\in S}\mathbb{Z}, \tag{29}$$

where

(1) $K_2(K(S))$ denotes the second Milnor K-group of the rational function field $K(S)$, and ∂ is the map whose "component at Z" is the tame symbol attached to the valuation ord_Z;

(2) the group
$$\Theta := \bigoplus_{Z\subset S} K(Z)^\times$$
is the set of finite formal linear combinations $\sum_i (Z_i,u_i)$, where the Z_i are irreducible curves in S and u_i is a rational function on Z_i;

(3) the map div is the divisor map and the direct sum defining its target is taken over all closed points $P\in S_K$.

Given a closed point $P\in X$ and a rational function u on X, an element of Θ of the form $(\{P\}\times X,u)$ (resp. of the form $(X\times\{P\},u)$) is said to be *vertical* (resp. *horizontal*). A linear combination of vertical and horizontal terms is said to be *negligible*. Similar definitions apply to the tensor product $\Theta\otimes F$ over any field F.

Let $\Delta\subset S$ be a copy of the curve X diagonally embedded in S. Let F denote the field introduced in §2.1.4, $u\in\mathcal{O}(Y_K)^\times\otimes F$ be a modular unit with coefficients in F, and consider the element $(\Delta,u)\in\Theta\otimes F$.

Lemma 3.1. *There exists a negligible element $\theta_u\in\Theta\otimes F$ satisfying*
$$\mathrm{div}(\theta_u) = \mathrm{div}(\Delta,u).$$

Proof. Let $D_u = \mathrm{div}(\Delta,u)\in\coprod_{P\in S}F$ be the image of the element $(\Delta,u)\in\Theta$ under the divisor map. Since D_u is an F-linear combination of elements of the form $(c_1,c_1)-(c_2,c_2)$ where c_1 and c_2 are cusps of the modular curve X_K, it is enough to construct a negligible element $\theta\in\Theta\otimes\mathbb{Q}$ satisfying

$$\mathrm{div}(\theta) = (\mathrm{c}_1,\mathrm{c}_1)-(\mathrm{c}_2,\mathrm{c}_2). \tag{30}$$

By the Manin-Drinfeld theorem, there is an element $\alpha\in\mathcal{O}(Y_K)^\times\otimes\mathbb{Q}$ whose divisor is c_1-c_2, and the negligible element given by
$$\theta = (\{c_1\}\times X,\alpha) + (X\times\{c_2\},\alpha)$$
satisfies (30). The lemma follows. □

Thanks to Lemma 3.1, we can associate to any element of the form $(\Delta,u)\in\Theta\otimes F$ the element

$$\Delta_u := \text{ class of } \quad (\Delta,u)-\theta_u \quad \text{ in } H^3_{\mathcal{M}}(S,F(2)). \tag{31}$$

These elements were introduced by Beilinson in [Bei, Ch. 2, §6]. A variant ([Fl, Prop 2.1]) of the above construction was later exploited by Flach in loc. cit. to prove the finiteness of the Tate-Shafarevic group of the symmetric square of an elliptic curve, using the method of Kolyvagin. We call Δ_u the *Beilinson-Flach element* attached to the modular unit $u \in \mathcal{O}(Y_K)^\times \otimes F$. Strictly speaking, Δ_u is not a well-defined element in $H^3_{\mathcal{M}}(S, F(2))$, as it is only well-defined modulo the F-vector space generated by the classes of negligible elements. However, this inherent ambiguity will not lead to problems because the image of Δ_u under the relevant piece of the regulator maps will turn out to depend only on u and not on the choice of θ_u made in defining Δ_u. See Proposition 3.3 below for more details.

3.2. **Complex regulators.** Fix an embedding of K into the field of complex numbers. Following the definitions in [Bei, §2], [DS, §2], the *complex regulator* on $H^3_{\mathcal{M}}(S_{\mathbb{C}}, \mathbb{Q}(2))$ may be regarded as a map

$$\mathbf{reg}_{\mathbb{C}} : H^3_{\mathcal{M}}(S_{\mathbb{C}}, \mathbb{Q}(2)) \longrightarrow (\mathrm{Fil}^1 H^2_{\mathrm{dR}}(S/\mathbb{C}))^\vee, \tag{32}$$

where here the superscript $\vee$ denotes the complex linear dual. It sends the class of $\theta = \sum_i (Z_i, u_i)$ to the element $\mathbf{reg}_{\mathbb{C}}(\theta)$ defined by

$$\mathbf{reg}_{\mathbb{C}}(\theta)(\omega) = \frac{1}{2\pi i} \sum_i \int_{Z_i - Z_i^{\mathrm{sing}}} \omega \log |u_i|.$$

Recall the modular unit u_χ associated to the Dirichlet character χ, and the class $\eta_f^{\mathrm{ah}} \in H^1_{\mathrm{dR}}(X/\mathbb{C})$ attached to the cusp form f. Moreover, write as customary $\omega_g \in \mathrm{Fil}^1 H^1_{\mathrm{dR}}(X/\mathbb{C})$ for the class associated to the regular differential $2\pi i g(z)dz$.

The tensor product $\omega_g \otimes \eta_f^{\mathrm{ah}}$ of these classes gives rise, via the Künneth decomposition of $H^2_{\mathrm{dR}}(S/\mathbb{C})$, to an element of $\mathrm{Fil}^1 H^2_{\mathrm{dR}}(S/\mathbb{C})$.

Proposition 3.2. *With notations as above, we have*

$$\mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\mathrm{ah}}) = (-2i)[\Gamma_0(N) : \Gamma_1(N)(\pm 1)] \langle f^*, f^* \rangle_{2,N}^{-1} \left\langle f^*(z), \log |u_\chi(z)| \cdot g(z) \right\rangle_{2,N}.$$

Proof. Since the differential $\omega_g \otimes \eta_f^{\mathrm{ah}}$ vanishes identically on the horizontal and vertical curves on S, the negligible element θ_{u_χ} arising in the definition of Δ_{u_χ} does not contribute to the value of the regulator at that class. Hence

$$\begin{aligned} \mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\mathrm{ah}}) &= \int_{X(\mathbb{C})} \frac{\bar{f}^*(z)}{\langle f^*, f^* \rangle_{2,N}} g(z) \log |u_\chi(z)| dz d\bar{z} \\ &= (-2i)[\Gamma_0(N) : \Gamma_1(N)(\pm 1)] \langle f^*, f^* \rangle_{2,N}^{-1} \left\langle f^*(z), \log |u_\chi(z)| \cdot g(z) \right\rangle_{2,N}, \end{aligned}$$

where the last equality follows from the explicit formula for the Petersson scalar product on $S_k^{\mathrm{ra}}(N, \chi_f^{-1})$. □

3.3. **p-adic regulators.** Let K_p be a finite extension of $\mathbb{Q}_p$ containing K and fix an embedding of K_p in $\mathbb{C}_p$. Write $\mathcal{O}_p$, resp. k_p for the ring of integers, resp. the residue field of K_p. Let $\mathcal{X}$ denote the (Deligne-Rapoport) smooth model of X over $\mathcal{O}_p$, and $\tilde{\mathcal{X}}/k_p$ its special fiber. Define $\mathcal{S} = \mathcal{X} \times \mathcal{X}$, which is a smooth projective model of S_{K_p} over $\mathcal{O}_p$.

In analogy with the complex regulator (32), there is a p-adic syntomic regulator map

$$\mathbf{reg}_p : H^3_{\mathcal{M}}(S_{K_p}, \mathbb{Q}(2)) \longrightarrow (\mathrm{Fil}^1 H^2_{\mathrm{dR}}(S/K_p))^\vee := \mathrm{Hom}(\mathrm{Fil}^1 H^2_{\mathrm{dR}}(S/K_p), K_p) \tag{33}$$

arising from the syntomic Chern character in K-theory (cf. [Gros], [Ni], [Bes3]).

After possibly enlarging the field K_p, let $\{P_1, ..., P_t\} \subset \mathcal{X}(\mathcal{O}_p)$ be a set of points consisting of the cusps and of a choice of a lift of every supersingular point in $\tilde{\mathcal{X}}(\bar{\mathbb{F}}_p)$. Set

$$\mathcal{X}' = \mathcal{X} \setminus \{P_1, ..., P_t\}, \qquad X' = \mathcal{X}' \times_{\mathrm{spec}\, \mathcal{O}_p} \mathrm{spec}\, K_p.$$

Let red : $\mathcal{X}(\mathcal{O}_p) \longrightarrow \tilde{\mathcal{X}}(k_p)$ denote the reduction map and let $\mathcal{A} \subset X(K_p)$ be the affinoid subspace of the rigid analytic variety underlying X defined by

$$\mathcal{A} := X(K_p) - \mathrm{red}^{-1}(\{\tilde{P}_1, \ldots, \tilde{P}_t\}), \qquad \tilde{P}_j := \mathrm{red}(P_j).$$

Fix a system $\{\mathcal{W}_\epsilon\}_{\epsilon>0}$ of wide open neighborhoods of $\mathcal{A}$ as in [DR, §2.1] and denote Φ the canonical lift of Frobenius on X as in [DR, §2.2]. As explained in loc. cit., restriction from X' to $\mathcal{W}_\epsilon$ gives rise to an isomorphism

$$H^1_{\mathrm{dR}}(X') \xrightarrow{\mathrm{comp}_\epsilon} H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon) \tag{34}$$

between the de Rham cohomology of the open curve X' and the rigid cohomology $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ of $\mathcal{W}_\epsilon$. The inclusion $X' \subset X$ yields by restriction a monomorphism

$$H^1_{\mathrm{dR}}(X) \hookrightarrow H^1_{\mathrm{dR}}(X'),$$

and the image of $H^1_{\mathrm{dR}}(X)$ under comp_ϵ consists of those classes in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ whose annular residues about all the points $\{P_i\}$ vanish. The lift Φ of Frobenius induces a linear endomorphism of $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ which preserves the subspace $H^1_{\mathrm{dR}}(X)$.

Label now two copies of X as X_1 and X_2, denote by Φ_1 and Φ_2 the corresponding canonical lifts of Frobenius on the system of wide open neighborhoods $\mathcal{W}_\epsilon$, and write $\Phi_{12} := (\Phi_1, \Phi_2)$ for the associated lift of Frobenius on the product $X_1 \times X_2$.

Choose a polynomial $P(x) \in \mathbb{C}_p[x]$ such that

(i) $P(\Phi_{12})$ annihilates the class of $\omega_g \otimes \frac{du_\chi}{u_\chi}$ in $H^2_{\mathrm{rig}}(\mathcal{W}^2_\epsilon)$;

(ii) $P(\Phi)$ is an invertible endomorphism on $H^1_{\mathrm{dR}}(X')$.

Such a polynomial exists, since the eigenvalues of Φ_{12} acting on the space spanned by the Frobenius translates of $\omega_g \otimes \frac{du_\chi}{u_\chi}$ have complex absolute value $p^{3/2}$, while Φ acts on $H^1_{\mathrm{dR}}(X')$ with eigenvalues of complex absolute value $p^{1/2}$ and p.

Thanks to (i), there exists a rigid analytic one-form

$$\varrho_P \in \Omega^1(\mathcal{W}^2_\epsilon) \text{ such that } d(\varrho_P) = P(\Phi_{12})\left(\omega_g \otimes \frac{du_\chi}{u_\chi}\right). \tag{35}$$

This form, which depends on the choice of P, is only determined up to *closed* forms in $\Omega^1(\mathcal{W}^2_\epsilon)$ by (35).

In order to adapt our calculations to Besser's in [Bes2] and [Bes3], it will be convenient to fix a particular choice of polynomial P and form ϱ_P. (In the next section we shall exploit the fact that the computations performed there hold independently of the choice of P, and will work with a different polynomial so that we can take advantadge of the results obtained in [DR].)

Let $P_g(t) \in \mathbb{C}_p[t]$ be a polynomial such that $P_g(\Phi)$ annihilates the class of ω_g in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$. Specifically, we may set $P_g(t) := t^2 - a_p(g)t + \chi_g(p)p$, and let $F_g \in \mathcal{O}_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ be a Coleman integral of ω_g, that is to say, a rigid analytic function such that

$$p dF_g = p\omega_{g^{[p]}} = P_g(\Phi)\omega_g \tag{36}$$

(cf. for example equation (127) of [DR]). Likewise, let $P_{E_\chi}(t) \in \mathbb{C}_p[t]$ be a polynomial such that $P_{E_\chi}(\Phi)$ annihilates the class of the Eisenstein series $E_\chi = \frac{du_\chi}{u_\chi}$ in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$. Here we make the specific choice $P_{E_\chi}(t) := t^h - p^h$, where h is the order of the root of unity $\chi(p)$ (in other words, Φ^h/p^h fixes the class of E_χ). Although a more optimal choice for $P_{E_\chi}(t)$ would have been the linear polynomial $t - \chi(p)p$, we made here a choice corresponding to the one made in the definition of the modified syntomic regulator $\mathrm{reg}(u_\chi)$ of the function u_χ (cf. [Bes2, Prop. 10.3]). The rigid analytic function

$$F_{E_\chi} := p^{-h} P_{E_\chi}(\Phi)\log(u_\chi) \in \mathcal{O}_{\mathrm{rig}}(\mathcal{W}_\epsilon) \tag{37}$$

is a Coleman integral of E_χ, satisfying

$$p^h dF_{E_\chi} = P_{E_\chi}(\Phi)E_\chi.$$

Given two choices as above of polynomials $P_g(t) = \prod_i (t-\alpha_i)$ and $P_{E_\chi}(t) = \prod_j (t-\beta_j)$, it is clear that the polynomial

$$P(t) := P_g(t) \star P_{E_\chi}(t) := \prod_{i,j} (t - \alpha_i\beta_j) \tag{38}$$

satisfies (i) above. Moreover, as explained in [Bes1, Lemma 4.2, (4)], there exist polynomials $a(t_1,t_2)$, $b(t_1,t_2)$ such that $P(t_1 \cdot t_2) = p^{-1}a(t_1,t_2)P_g(t_1) + p^{-h}b(t_1,t_2)P_{E_\chi}(t_2) \in \mathbb{C}_p[t_1,t_2]$ and one checks that

$$\varrho_P \quad = \quad a(\Phi_1,\Phi_2)\left(F_g \otimes \frac{du_\chi}{u_\chi}\right) \quad + \quad b(\Phi_1,\Phi_2)(\omega_g \otimes F_{E_\chi}) \quad \in \quad \Omega^1(\mathcal{W}_\epsilon^2) \tag{39}$$

then satisfies (35).

There is a certain degree of ambiguity in (39): neither the Coleman primitives F_g, F_{E_χ} nor the polynomials $a(t_1,t_2)$, $b(t_1,t_2)$ are unique. But all solutions of the differential equation (35) are of the form (39); moreover, given one such ϱ_P, all them can be written as $\varrho_P + \varrho_0$ with ϱ_0 a closed 1-form on $\mathcal{W}_\epsilon^2$.

We can single out a canonical choice of ϱ_P (up to exact 1-forms on $\mathcal{W}_\epsilon^2$) by setting $F_g(\infty) = F_{E_\chi}(\infty) = 0$ in (39); more precisely, in doing this, two different choices of pairs $(a(t_1,t_2), b(t_1,t_2))$, $(a'(t_1,t_2), b'(t_1,t_2))$ allowed by [Bes1, Lemma 4.2, (4)] give rise to forms $\varrho_{P,a,b}$, $\varrho_{P,a',b'}$ such that $\varrho_0 = \varrho_{P,a,b} - \varrho_{P,a',b'}$ is exact on $\mathcal{W}_\epsilon^2$ and therefore the class of ϱ_0 in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon^2)$ vanishes.

Imposing $F_g(\infty) = 0$ amounts to normalizing the q-expansion of F_g to be

$$F_g(q) = \sum_{p\nmid n} \frac{a_n(g)}{n} q^n, \tag{40}$$

and the condition $F_{E_\chi}(\infty) = 0$ is equivalent to normalizing the modular unit u_χ as was done in Definition 2.4. This way F_{E_χ} also equals the modified syntomic regulator $\mathrm{reg}(u_\chi)$ of u_χ defined in [Bes2, Prop. 10.3].

Let $\Delta \subset \mathcal{W}_\epsilon^2$ denote the diagonal and define

$$\xi'_P := [\varrho_{P|\Delta}] \in H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon) \simeq H^1_{\mathrm{dR}}(X'). \tag{41}$$

The above discussion shows that the class ξ'_P in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon) = \frac{\Omega^1(\mathcal{W}_\epsilon)}{d\mathcal{O}(\mathcal{W}_\epsilon)}$ is well-defined. Moreover, in view of condition (ii), we can now set

$$\xi' := P(\Phi)^{-1} \cdot \xi'_P \in H^1_{\mathrm{dR}}(X'), \tag{42}$$

which is directly seen to be independent of the choice of P.

Finally, let $\mathrm{spl}_X : H^1_{\mathrm{dR}}(X') \longrightarrow H^1_{\mathrm{dR}}(X)$ denote the Frobenius equivariant splitting of the short exact sequence

$$0 \to H^1_{\mathrm{dR}}(X) \longrightarrow H^1_{\mathrm{dR}}(X') \longrightarrow K_p(-1)^{t-1} \to 0 \tag{43}$$

and set $\xi := \mathrm{spl}_X(\xi') \in H^1_{\mathrm{dR}}(X)$.

Proposition 3.3. *With notations as above, we have*

$$\mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\mathrm{ur}}) = \langle \eta_f^{\mathrm{ur}}, \xi \rangle,$$

where $\langle\ ,\ \rangle$ *is the pairing on* $H^1_{\mathrm{dR}}(X)$ *induced by Poincaré duality.*

Proof. Thanks to the work of Besser [Bes3], the p-adic syntomic regulator (33) admits the following description in terms of Coleman integration. Let $\theta = \sum_i (Z_i, u_i)$ be an element in $K_1^{(2)}(S)$ and write $\iota_i : Z_i \hookrightarrow S$ for the embedding of Z_i in S given by inclusion. Assume for simplicity that the curves Z_i are all non-singular, and that θ is *integral*, by what we mean that for each i:

- the curve Z_i admits a smooth integral model $\mathcal{Z}_i$ over $\mathcal{O}_p$, and
- the divisor of u_i, when regarded as a function on $\mathcal{Z}_i$, does not contain the special fiber.

Note that these conditions are satisfied in our setting.

Under this assumption, θ lies in the image of the natural restriction map $K_1(\mathcal{S}) \longrightarrow K_1(S)$.

Let $\Omega^{\mathrm{II}}(X_{K_p})$ denote the space of differential forms of the second kind on X_{K_p}, that is to say, the space of meromorphic 1-forms whose residue at any point of the curve is zero. There is an exact sequence

$$0 \to K_p(X)^\times \xrightarrow{d} \Omega^{\mathrm{II}}(X_{K_p}) \longrightarrow H^1_{\mathrm{dR}}(X/K_p) \to 0$$

and for any $\eta \in \Omega^{\mathrm{II}}(X_{K_p})$ we write $[\eta]$ for its class in $H^1_{\mathrm{dR}}(X/K_p)$.

Instead of invoking the description of the p-adic syntomic regulator in terms of Besser-de Jeu's global triple index as stated in the main theorem of [Bes3], it will be more convenient for us to exploit [Bes3, Prop. 6.3], which provides a formula for (33) in the language of Besser's finite polynomial cohomology [Bes1]. In order to state this formula, let H^*_{ms} and H^*_{fp} denote, respectively, Besser's modified version of syntomic cohomology and finite polynomial cohomology: cf. e.g. [Bes3, §2] for a quick review of both and their interactions.

Let $\omega \in \Omega^1(X_{K_p})$ be a regular form on X and $\eta \in \Omega^{\mathrm{II}}(X_{K_p})$ be a differential of the second kind, regular on some affine curve $X^0 \subset X$. Write

$$\omega_1 = \pi_1^*(\omega) \in \Omega^1(S), \qquad \eta_2 = \pi_2^*(\eta) \in \Omega^{\mathrm{II}}(S)$$

for the pull-back of ω and η under the projection of S into the first and second component, respectively.

Then the class $\omega_1 \wedge [\eta_2]$ is an element of $\mathrm{Fil}^1 H^2_{\mathrm{dR}}(S)$ and, according to [Bes3, Theorem 1.1, Proposition 6.3]:

$$\mathbf{reg}_p(\theta)(\omega_1 \otimes [\eta_2]) = \sum_i \langle \iota_i^* \tilde{\eta}_2, \iota_i^* \tilde{\omega}_1 \cup \mathrm{reg}(u_i) \rangle_{\mathcal{Z}_i^0, \mathrm{fp}}, \tag{44}$$

where

- $Z_i^0 = Z_i \cap (X \times X^0)$, $\mathcal{Z}_i^0$ is the model for Z_i^0 deduced from $\mathcal{Z}_i$,
- $\mathrm{reg}(u_i) \in H^1_{\mathrm{ms}}(\mathcal{Z}_i^0, 1) \subseteq H^1_{\mathrm{fp}}(\mathcal{Z}_i^0, 1, 2)$ is the regulator of the function u_i as defined in [Bes2, Prop. 10.3],
- $\iota_i^* \tilde{\omega}_1 \in H^1_{\mathrm{fp}}(\mathcal{Z}_i, 1, 1)$ is a Coleman primitive of $\iota_i^* \omega \in \Omega^1(Z_i)$,
- $\iota_i^* \tilde{\eta}_2 \in H^1_{\mathrm{fp},c}(\mathcal{Z}_i^0, 0, 1)$ is the single lift of $\iota_i^*([\eta_2])$ under the isomorphism

$$\mathrm{p} : H^1_{\mathrm{fp},c}(\mathcal{Z}_i^0, 0, 1) \xrightarrow{\sim} H^1_{\mathrm{dR}}(Z_i) \tag{45}$$

of [Bes3, Lemma 6.2], and

$$\langle\, , \rangle_{\mathcal{Z}_i^0, \mathrm{fp}} : H^1_{\mathrm{fp},c}(\mathcal{Z}_i^0, 0, 1) \times H^2_{\mathrm{fp}}(\mathcal{Z}_i^0, 2, 3) \longrightarrow H^3_{\mathrm{fp},c}(\mathcal{Z}_i^0, 2, 4) \simeq H^2_{\mathrm{dR},c}(Z_i^0) \overset{\mathrm{tr}}{\simeq} K_p \tag{46}$$

is the pairing induced by Poincaré duality in finite polynomial cohomology. Here $H^*_{\mathrm{fp},c}$ stands for finite polynomial cohomology with compact support, as introduced in [Bes3, §4]. The cup-product (46) is constructed in loc. cit., where it is also shown that it satisfies the projection formula.

At the time [Bes3] was written, the results were subject to the compatibility of pushforward maps in syntomic and motivic cohomology, as specified in [Bes3, Conjecture 4.2]. At present this compatibility has been checked by Déglise and Mazzari [DM], and thus (44) holds unconditionally.

Let us now apply (44) to the Beilinson-Flach element Δ_{u_χ} that was introduced in (31). Recall that the curves in $X \times X$ on which Δ_{u_χ} is supported are the images of X under the diagonal embedding $\iota_{12}(x) = (x, x)$ and the various horizontal and vertical embeddings $\iota_{1,c}(x) = (x, c)$ and $\iota_{2,c}(x) = (c, x)$, where c is a cusp on the modular curve X.

We firstly claim that the terms on the right-hand side of (44) corresponding to $\iota_{1,c}$ and $\iota_{2,c}$ vanish and the one corresponding to ι_{12} is independent of the choices of lifts to finite polynomial cohomology.

To see that, put $\omega = \omega_g$ and $\eta = \eta_f^{\mathrm{ur}}$ and recall $X' = \mathcal{X}' \times K_p$ is the curve obtained from X by removing a finite set of points including all the cusps. Note first that $\iota_{1,c}^*([\eta_2]) = 0 \in H^1_{\mathrm{dR}}(X)$, because the composition $\pi_2 \circ \iota_{1,c}$ is the constant function c on X. Hence, since the map p in (45) is an isomorphism, the class of the lift $\iota_{1,c}^*(\tilde\eta_2)$ is also trivial and

$$\langle \iota_{1,c}^* \tilde\eta_2, \iota_{1,c}^* \tilde\omega_1 \cup \mathrm{reg}(u)\rangle_{\mathcal{X}',\mathrm{fp}} = 0,$$

for any rational function u.

We similarly have $\iota_{2,c}^*(\omega_1) = 0 \in \Omega^1(X)$ because $\pi_1 \circ \iota_{2,c} = c$. Notice however that a lift of 0 to $H^1_{\mathrm{fp}}(\mathcal{X}, 1, 1)$ is not necessarily trivial, but represented by a pair in $\mathcal{O}_{\mathrm{rig}}(\mathcal{W}_\epsilon) \oplus \Omega^1(X)$ of the form $[(\lambda, 0)]$, where λ is a constant. Then, if u is a modular unit on X, the cup-product

$$\iota_{2,c}^* \tilde\omega_1 \cup \mathrm{reg}(u) \in H^2_{\mathrm{fp}}(\mathcal{X}', 2, 3) \simeq H^1_{\mathrm{dR}}(X') \simeq H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$$

may be represented by the pair $(\lambda \frac{du}{u}_{|\mathcal{W}_\epsilon}, 0)$. But then

$$\langle \iota_{2,c}^* \tilde\eta_2, \iota_{2,c}^* \tilde\omega_1 \cup \mathrm{reg}(u)\rangle_{\mathcal{X}',\mathrm{fp}} = \lambda \langle \eta_f^{\mathrm{ur}}, \frac{du}{u}\rangle_{\mathrm{dR}} = 0 \tag{47}$$

because the cusp form f is orthogonal to the Eisenstein series $\frac{du}{u}$. This accounts for the vanishing of the horizontal and vertical terms, and explains why we call them negligible.

As for the diagonal term, let us show that $\langle \iota_{12}^* \tilde\eta_2, \iota_{12}^* \tilde\omega_1 \cup \mathrm{reg}(u_\chi)\rangle_{\mathcal{X}',\mathrm{fp}}$ is independent of the choices of lifts to finite polynomial cohomology. Since $\pi_1 \circ \iota_{12}$ and $\pi_2 \circ \iota_{12}$ are both the identity map on X, this is just $\langle \tilde\eta_f^{\mathrm{ur}}, \tilde\omega_g \cup \mathrm{reg}(u_\chi)\rangle_{\mathcal{X}',\mathrm{fp}}$. Again there is a single choice for $\tilde\eta_f^{\mathrm{ur}}$, but the Coleman integral F_g of ω_g is only well-defined up to a constant. The difference between any two choices is then equal to

$$\langle \tilde\eta_f^{\mathrm{ur}}, [(\lambda, 0)] \cup \mathrm{reg}(u_\chi)\rangle_{\mathcal{X}',\mathrm{fp}} = \lambda \left\langle \eta_f^{\mathrm{ur}}, \frac{du_\chi}{u_\chi} \right\rangle_{\mathrm{dR}}$$

for some $\lambda \in K_p$, and the same orthogonality argument between cusp and Eisenstein forms again shows that this is 0. The claim follows.

Summing up, we obtain from (44) that

$$\mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\mathrm{ur}}) = \langle \tilde\eta_f^{\mathrm{ur}}, \tilde\omega_g \cup \mathrm{reg}(u_\chi)\rangle_{\mathcal{X}',\mathrm{fp}}. \tag{48}$$

Recall that $\tilde\omega_g$ may be represented by the pair (F_g, ω_g) where $F_g \in \mathcal{O}_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ is a Coleman integral of ω_g, which in light of the above claim we are entitled to normalize as it was done in (40). Besides, by [Bes2, Prop. 10.3] the class $\mathrm{reg}(u_\chi)$ is represented by the pair $(F_{E_\chi}, \frac{du_\chi}{u_\chi}) \in \mathcal{O}_{\mathrm{rig}}(\mathcal{W}_\epsilon) \oplus \Omega^1(X')$ where F_{E_χ} is the Coleman integral of $\frac{du_\chi}{u_\chi}$ introduced in (37) and normalized as we explained right after (40).

By definition, $\tilde\omega_g \cup \mathrm{reg}(u_\chi)$ is the restriction to the diagonal of $\pi_1^* \tilde\omega_g \wedge \pi_2^* \mathrm{reg}(u_\chi)$. Note that the polynomial P defined in equation (38) satisfies the properties (i) and (ii) above. The class

$\pi_1^*\tilde{\omega}_g \wedge \pi_2^*\mathrm{reg}(u_\chi)$ in $H^2_{\mathrm{fp}}(\mathcal{X}'^2, 2, 3)$ may then be represented by the pair

$$(\varrho_P, \pi_1^*\omega_g \wedge \pi_2^*\frac{du_\chi}{u_\chi}) \in \Omega^1_{\mathrm{rig}}(\mathcal{W}^2_\epsilon) \oplus \Omega^2(X'^2) \tag{49}$$

where ϱ_P is the form introduced in (39), which satisfies

$$d\varrho_P = P(\Phi_{12})(\pi_1^*\omega_g \wedge \pi_2^*\frac{du_\chi}{u_\chi}). \tag{50}$$

Let us again remark that this differential equation does not determine ϱ_P uniquely, but that the above normalizations of F_g and F_{E_χ} completely determine it up to exact 1-forms on $\mathcal{W}^2_\epsilon$. Obviously, when we restrict (49) to the diagonal, this ambiguity does not affect the class we obtain in $H^2_{\mathrm{fp}}(\mathcal{X}', 2, 3)$, because exact 1-forms on $\mathcal{W}_\epsilon$ vanish in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$.

In conclusion, the class $\tilde{\omega}_g \cup \mathrm{reg}(u_\chi)$ in $H^2_{\mathrm{fp}}(\mathcal{X}', 2, 3)$ may be represented by the pair

$$(\iota_{12}^*(\varrho_P), 0) \in \Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon) \oplus \Omega^2(X'),$$

where ϱ_P is as above and $\iota_{12}^*(\varrho_P)$ is the form denoted ξ'_P in (41).

As in [Bes1, (14)] there is a commutative diagram

$$\begin{array}{ccc} H^1_{\mathrm{fp},c}(\mathcal{X}', 0, 1) \times H^1_{\mathrm{dR}}(X') & \xrightarrow{\mathrm{Id}\times \mathrm{i}} & H^1_{\mathrm{fp},c}(\mathcal{X}', 0, 1) \times H^2_{\mathrm{fp}}(\mathcal{X}', 2, 3) \\ \downarrow{\scriptstyle \mathrm{p}\times\mathrm{Id}} & & \downarrow{\scriptstyle \langle\,,\rangle_{\mathrm{fp}}} \\ H^1_{\mathrm{dR},c}(X')^{w=1} \times H^1_{\mathrm{dR}}(X') & \xrightarrow{\langle\,,\rangle_{\mathrm{dR}}} & H^2_{\mathrm{dR},c}(X') \simeq H^3_{\mathrm{fp},c}(\mathcal{X}', 2, 4), \end{array} \tag{51}$$

where $H^1_{\mathrm{dR},c}(X')^{w=1}$ stands for the pure submodule of weight 1 of $H^1_{\mathrm{dR},c}(X')$. In fact both maps

$$H^1_{\mathrm{dR}}(X') \xrightarrow{\mathrm{i}} H^1_{\mathrm{fp}}(\mathcal{X}', 2, 3) \text{ and } H^1_{\mathrm{fp},c}(\mathcal{X}', 0, 1) \xrightarrow{\mathrm{p}} H^1_{\mathrm{dR},c}(X')^{w=1}$$

are isomorphisms, as it follows from [Bes3, (2.7) and the first assertion of Lemma 2.8].

By definition of i, the preimage of $\tilde{\omega}_g \cup \mathrm{reg}(u_\chi) = [(\xi'_P, 0)]$ under i is the class in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ of the 1-form $P(\Phi)^{-1}(\xi'_Q) = \xi'$. To conclude, we now deduce from the commutativity of (51) that

$$\langle \eta_f^{\mathrm{ur}}, \xi' \rangle_{\mathrm{dR}} = \langle \tilde{\eta}_f^{\mathrm{ur}}, \mathrm{i}(\xi') \rangle_{\mathrm{fp}} = \langle \tilde{\eta}_f^{\mathrm{ur}}, \tilde{\omega}_g \cup \mathrm{reg}(u_\chi) \rangle_{\mathrm{fp}}.$$

Since the class η_f^{ur} is orthogonal to the complement of $H^1_{\mathrm{dR}}(X)$ in $H^1_{\mathrm{dR}}(X')$ under the Frobenius equivariant splitting of (43), we have $\langle \eta_f^{\mathrm{ur}}, \xi' \rangle_{\mathrm{dR}} = \langle \eta_f^{\mathrm{ur}}, \xi \rangle_{\mathrm{dR}}$ and the proposition follows. □

4. The Beilinson formula

Let $f \in S_2(N, \chi_f)$, $g \in S_2(N, \chi_g)$ be eigenforms of weight 2 as in Section 2.1.4. Recall that f and g are not assumed to be newforms. Moreover, we insist on the condition $\chi_f \neq \chi_g^{-1}$, which implies that $\chi = \chi_f^{-1}\chi_g^{-1}$ is non-trivial.

4.1. **The complex setting.** In [Bei, Ch. 2, § 6], Beilinson relates the image of Δ_{u_χ} under the complex regulator map to the value at $s = 2$ of the Rankin L-series attached to $f \otimes g$. The following explicit version of Beilinson's theorem is a slight generalisation of the results of [BaSr].

Proposition 4.1. *For cusp forms f and g of weight two as in Section 2.1.4, we have*

$$\frac{L(f \otimes g, 2)}{\langle f^*, f^* \rangle_{2,N}} = (8i)\pi^3[\Gamma_0(N) : \Gamma_1(N)(\pm 1)]^{-1}N^{-2}\tau(\chi^{-1})\mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\mathrm{ah}}).$$

Proof. This follows by combining the explicit formula for $L(f \otimes g, 2)$ obtained in Proposition 2.5 with the explicit expression for $\mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})$ given in Proposition 3.2. □

4.2. **The p-adic setting.** Let $p \geq 3$ be a prime which does not divide N. Assume that f is ordinary at p (with respect to a fixed embedding of the field K_f in $\mathbb{C}_p$). Let $\mathbf{f}$ be the Hida family whose specialisation in weight 2 is the p-stabilisations of f, and let $L_p(\mathbf{f}, g)(k, j)$ be the p-adic L-function defined in Section 2.2.2.

Let $\mathcal{E}(f)$, $\mathcal{E}^*(f)$ and $\mathcal{E}(f, g, 2)$ be the p-adic multipliers defined in Sections 2.2.1 and 2.2.2. Recall that

$$\begin{aligned}\mathcal{E}(f,g,2) &= (1-\beta_p(f)\alpha_p(g)p^{-2})(1-\beta_p(f)\beta_p(g)p^{-2}) \\ &\quad \times(1-\beta_p(f)\alpha_p(g)\chi(p)p^{-1})(1-\beta_p(f)\beta_p(g)\chi(p)p^{-1}).\end{aligned}$$

The following p-adic Beilinson formula is the main result of this paper.

Theorem 4.2. *For cusp forms f and g of weight two as in Section 2.1.4, we have*

$$L_p(\mathbf{f}, g)(2,2) = \frac{\mathcal{E}(f,g,2)}{\mathcal{E}(f)\cdot\mathcal{E}^*(f)} \times \mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\mathrm{ur}}).$$

Proof. By the description of the p-adic L-function given in equation (28),

$$L_p(\mathbf{f}, g)(k,j) = \frac{1}{\mathcal{E}^*(f_k)} \left\langle \eta_{f_k}^{\mathrm{ur}}, \Xi(f_k, g, j)^{\mathrm{ord},p} \right\rangle_{k,X}$$

for all triples (k, ℓ, j) belonging to the set (26). Since the terms in the above expression vary analytically, taking the limit to $k = \ell = j = 2$ yields, in light of equation (25),

$$L_p(\mathbf{f}, g)(2,2) = \frac{1}{\mathcal{E}^*(f)} \left\langle \eta_f^{\mathrm{ur}}, e_{\mathrm{ord}}(d^{-1}E_{2,\chi}^{[p]} \cdot g) \right\rangle_{2,X}. \tag{52}$$

On the other hand, by Proposition 3.3

$$\mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\mathrm{ur}}) = \langle \eta_f^{\mathrm{ur}}, \xi \rangle.$$

Since

$$\Phi(\eta_f^{\mathrm{ur}}) = \alpha_p(f)\eta_f^{\mathrm{ur}}, \qquad \langle \Phi(\eta_f^{\mathrm{ur}}), \Phi(\xi)\rangle = p\langle \eta_f^{\mathrm{ur}}, \xi\rangle, \qquad \alpha_p(f)\beta_p(f) = \chi_f(p)p,$$

we deduce by multi-linearity that

$$\langle \eta_f^{\mathrm{ur}}, \xi \rangle = P\big(\chi_f^{-1}(p)\beta_p(f)\big)^{-1} \langle \eta_f^{\mathrm{ur}}, \xi'_P \rangle.$$

Since f is an ordinary eigenform, the quantity $\langle \eta_f^{\mathrm{ur}}, \xi'_P\rangle$ only depends on the f^*-isotypical ordinary projection of ξ'_P, that is to say, $\langle \eta_f^{\mathrm{ur}}, \xi'_P\rangle = \langle \eta_f^{\mathrm{ur}}, e_{f^*}e_{\mathrm{ord}}\xi'_P\rangle$.

Choose the polynomial $P(x)$ satisfying conditions (i) and (ii) to be

$$P(x) := (x - \alpha_p(g)) \cdot (x - \alpha_p(g)\chi(p)p) \cdot (x - \beta_p(g)) \cdot (x - \beta_p(g)\chi(p)p).$$

This choice of P has the advantage of allowing us to directly invoke the calculations already performed in [DR, Prop. 5.4]. They give

$$e_{f^*}e_{\mathrm{ord}}\xi'_P = \chi_f(p)^{-2}p^4\mathcal{E}(f) \cdot e_{f^*}e_{\mathrm{ord}}(d^{-1}E_{2,\chi}^{[p]} \cdot g).$$

A direct calculation shows that

$$\mathcal{E}(f,g,2) = p^{-4}\chi_f(p)^{-2}P\big(\chi_f^{-1}(p)\beta_p(f)\big).$$

By combining the above remarks, we find the following expression for the p-adic regulator:

$$\mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\mathrm{ur}}) = \frac{\mathcal{E}(f)}{\mathcal{E}(f,g,2)} \times \left\langle \eta_f^{\mathrm{ur}}, e_{f^*}e_{\mathrm{ord}}(d^{-1}E_{2,\chi}^{[p]} \cdot g)\right\rangle_{2,X}. \tag{53}$$

The theorem follows by comparing equations (52) and (53). □

Remark 4.3. Note that the modular form g that arises in Theorem 4.2 is fixed throughout the argument, and is thus not required to be ordinary at p.

Assume now that both f and g are ordinary at p (with respect to a fixed embedding of the field K_{fg} in $\mathbb{C}_p$). Let $\mathbf{f}$ and $\mathbf{g}$ be the Hida families whose specialisations in weight 2 are the p-stabilisations of f and g, respectively, and let $L_p(\mathbf{f}, \mathbf{g})(k, \ell, j)$ be the p-adic L-function defined in Section 2.2.2. The following corollary is an immediate consequence of Theorem 4.2.

Corollary 4.4. *For cusp forms f and g of weight two as in Section 2.1.4, we have*

$$L_p(\mathbf{f}, \mathbf{g})(2, 2, 2) = \frac{\mathcal{E}(f, g, 2)}{\mathcal{E}(f) \cdot \mathcal{E}^*(f)} \times \mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\mathrm{ur}}).$$

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