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Quaternions, polarizations and class numbers

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Abstract. We study abelian varieties A with multiplication by a totally indefinite quaternion algebra over a totally real number field and give a criterion for the existence of principal polarizations on them in pure arithmetic terms. Moreover, we give an expression for the number $\pi_0(A)$ of isomorphism classes of principal polarizations on A in terms of relative class numbers of CM fields by means of Eichler's theory of optimal embeddings. As a consequence, we exhibit simple abelian varieties of any even dimension admitting arbitrarily many non-isomorphic principal polarizations. On the other hand, we prove that $\pi_0(A)$ is uniformly bounded for simple abelian varieties of odd square-free dimension.

1. Introduction

It is well-known that every elliptic curve E over an arbitrary algebraically closed field admits a unique principal polarization up to translations. This is in general no longer shared by higher dimensional abelian varieties and it is a delicate question to decide whether a given abelian variety A is principally polarizable. Even, if this is the case, it is an interesting problem to investigate the set $\Pi_0(A)$ of isomorphism classes of principal polarizations on A. By a theorem of Narasimhan and Nori (cf. [22]), $\Pi_0(A)$ is a finite set. We shall denote its cardinality by $\pi_0(A)$.

The aim of this paper is to study these questions on abelian varieties with quaternionic multiplication. It will be made apparent that the geometric properties of these abelian varieties are encoded in the arithmetic of their ring of endomorphisms. The results of this paper shed some light on the geometry and arithmetic of the Shimura varieties that occur as moduli spaces of abelian varieties with quaternionic multiplication and their groups of automorphisms. In this regard, we refer the reader to [27] and [28]. Our results are also the basis of a study of the diophantine properties of abelian surfaces with quaternionic multiplication over number fields carried by Dieulefait and the author in [4].

Let us remark that a generic principally polarizable abelian variety admits a single class of principal polarizations. In ([14]), Humbert was the first to exhibit simple complex abelian surfaces with two non-isomorphic principal polarizations on them.

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Later, Hayashida and Nishi ([9] and [8]) computed $\pi_0(E_1 \times E_2)$ for isogenous elliptic curves $E_1/\mathbb{C}$ and $E_2/\mathbb{C}$ with complex multiplication. In positive characteristic, Ibukiyama, Katsura and Oort ([15]) related the number of principal polarizations on the power E^n of a supersingular elliptic curve to the class number of certain hermitian forms. With similar methods, Lange ([17]) produced examples of simple abelian varieties of high dimension with several principal polarizations. However, he showed that for an abelian variety with endomorphism algebra $\mathrm{End}(A) \otimes \mathbb{Q} = F$, a totally real number field, the number $\pi_0(A)$ is uniformly bounded in terms of the dimension of A: $\pi_0(A) \leq 2^{\dim(A)-1}$. That is, abelian varieties whose ring of endomorphisms is an order in a totally real number field admit several but not *arbitrarily many* principal polarizations.

It could be expected that Lange's or some other bound for $\pi_0(A)$ held for any simple abelian variety. Hence the question: *given $g \geq 1$, are there simple abelian varieties of dimension g with arbitrarily many non-isomorphic principal polarizations?*

As was already observed, this is not the case in dimension 1. In $g = 2$, only simple abelian surfaces with at most $\pi_0(A) = 2$ were known. One of our main results, stated in a particular case, is the following.

Theorem 1.1. *Let F be a totally real number field of degree $[F : \mathbb{Q}] = n$, let R_F denote its ring of integers and $\vartheta_{F/\mathbb{Q}}$ the different of F over $\mathbb{Q}$. Let A be a complex abelian variety of dimension $2n$ whose ring of endomorphisms $\mathrm{End}(A) \simeq \mathcal{O}$ is a maximal order in a totally indefinite quaternion division algebra B over F.*

Assume that the narrow class number $h_+(F)$ of F is 1 and that $\vartheta_{F/\mathbb{Q}}$ and $\mathrm{disc}(\mathcal{O})$ are coprime ideals. Then,

(1) *A is principally polarizable.*
(2) *The number of isomorphism classes of principal polarizations on A is*

$$\pi_0(A) = \frac{1}{2} \sum_S h(S),$$

where S runs among the set of orders in the CM-field $F(\sqrt{-D})$ that contain $R_F[\sqrt{-D}]$, the element $D \in F_+^*$ is taken to be a totally positive generator of the reduced discriminant ideal $\mathcal{D}_{\mathcal{O}}$ of $\mathcal{O}$ and $h(S)$ denotes its class number.

In particular, if A is an abelian surface,

$$\pi_0(A) = \begin{cases} \dfrac{h(-4D) + h(-D)}{2} & \text{if } D \equiv 3 \mod 4, \\ \dfrac{h(-4D)}{2} & \text{otherwise.} \end{cases}$$

We prove Theorem 1.1 in the more general form of Proposition 6.2 and our main Theorem 7.1. In order to accomplish it, we present an approach to the problem which

stems from Shimura's classical work [29] on analytic families of abelian varieties with prescribed endomorphism ring.

Our approach is essentially different from Lange's in [17] or Ibukiyama-Katsura-Oort's in [15]. Indeed, whereas in [17] and [15] the (noncanonical) interpretation of line bundles as symmetric endomorphisms is exploited, we translate the questions we are concerned with to Eichler's language of optimal embeddings. This leads us to solve a problem that has its roots in the work of O'Connor, Pall and Pollack (cf. [25]) and that has its own interest: see Section 4 for details.

In regard to the question above, the second main result of this article is the following.

Theorem 1.2. *Let g be a positive integer. Then*

(1) *If g is even, there exist simple abelian varieties A of dimension g such that $\pi_0(A)$ is arbitrarily large.*
(2) *If g is odd and square-free, $\pi_0(A) \leq 2^{g-1}$ for any simple abelian variety A of dimension g over $\mathbb{C}$.*

The boundless growth of $\pi_0(A)$ when g is even follows from our main Theorem 7.1 combined with analytical results on the asymptotic behaviour and explicit bounds for relative class numbers of CM-fields due to Horie-Horie ([12]) and Louboutin ([19], [20]). The second part of Theorem 1.2 follows from the ideas of Lange in [17]. See Section 8 for details.

The following corollary follows from Theorem 1.2 and the fact that any simple principally polarized abelian surface is the Jacobian of a smooth curve of genus 2 which, by Torelli's Theorem, is unique up to isomorphism.

Corollary 1.3. *There are arbitrarily large sets C_1 , ..., C_N of pairwise nonisomorphic genus 2 curves with isomorphic simple unpolarized Jacobian varieties $J(C_1) \cong J(C_2) \cong ... \cong J(C_N)$.*

In view of Theorem 1.2, it is natural to wonder whether there exist arbitrarily large sets of pairwise nonisomorphic curves of given even genus $g \geq 4$ with isomorphic unpolarized Jacobian varieties. In this direction, Ciliberto and van der Geer ([3]) proved the existence of two nonisomorphic curves of genus 4 whit isomorphic Jacobian varieties. Explicit examples of curves with isomorphic (nonsimple) Jacobians have been constructed by Howe ([13]), while examples of pairs of distinct modular curves of genus 2 defined over $\mathbb{Q}$ with isomorphic unpolarized absolutely simple Jacobian varieties have been obtained by González, Guàrdia and the author in [16].

Finally, let us note that the statement of Theorem 1.2 does not cover odd non square-free dimensions.

Conjecture 1.4. *Let g be a non square-free positive integer. Then there exist simple abelian varieties of dimension g such that $\pi_0(A)$ is arbitrarily large.*

The conjecture is motivated by the fact that, when g is odd and non square-free, there exist abelian varieties whose ring of endomorphisms is an order in a non commutative division algebra over a CM-field and there is a strong similitude between the arithmetic of the Néron-Severi groups of these abelian varieties and those in the quaternion case.

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2. Abelian varieties with quaternionic multiplication and their Néron-Severi group

Let F be a totally real number field of degree $[F:\mathbb{Q}]=n$ and let R_F be its ring of integers. Let B denote a totally indefinite division quaternion algebra over F and let $\mathcal{D}=\mathrm{disc}(B)=\prod_{i=1}^{2r}\wp_i$, where $\wp_i$ are finite prime ideals of F and $r\geq 1$, be its (reduced) discriminant ideal. We shall denote $\mathrm{n}=\mathrm{n}_{B/F}$ and $\mathrm{tr}=\mathrm{tr}_{B/F}$ the reduced norm and trace of B over F, respectively. Since B is totally indefinite, the Hasse-Schilling-Maass Norm Theorem asserts that $n(B)=F$ (cf. [10] and [31], p. 80). We fix an isomorphism of F-algebras

$$(\eta_\sigma): B\otimes_{\mathbb{Q}}\mathbb{R}\simeq\bigoplus_{\sigma}\mathrm{M}_2(\mathbb{R}^\sigma),$$

where $\sigma: F\hookrightarrow\mathbb{R}$ runs through the set of embeddings of F into $\mathbb{R}$ and $\mathbb{R}^\sigma$ denotes $\mathbb{R}$ as a F-vector space via the immersion σ. For any $\beta\in B$, we will often abbreviate $\beta^\sigma=\eta_\sigma(\beta)\in\mathrm{M}_2(\mathbb{R})$.

Let $\mathcal{O}$ be an hereditary order of B, that is, an order of B all whose one-sided ideals are projective. The discriminant ideal $\mathcal{D}_{\mathcal{O}}=\mathrm{disc}(\mathcal{O})$ of $\mathcal{O}$ is square-free and it can be written as $\mathcal{D}\cdot\mathcal{N}$ for some ideal $\mathcal{N}$ coprime to $\mathcal{D}$, which is called the *level* of $\mathcal{O}$ ([26], Chapter 9).

Definition 2.1. An abelian variety A has quaternionic multiplication by $\mathcal{O}$ if $\dim(A)=2n$ and $\mathrm{End}(A)\simeq\mathcal{O}$.

For the rest of this section, let $A/\mathbb{C}$ denote a complex abelian variety with quaternionic multiplication by the hereditary order $\mathcal{O}$. We will identify $\mathrm{End}(A)=\mathcal{O}$ and $\mathrm{End}(A)\otimes\mathbb{Q}=B$. Since B is a division algebra, A is simple, that is, it contains no proper sub-abelian varieties.

As a complex manifold, $A(\mathbb{C})=V/\Lambda$ for V a complex vector space of dimension g and $\Lambda\subset V$ a co-compact lattice that can be identified with the first group of integral singular homology $H_1(A,\mathbb{Z})$. The lattice Λ is naturally a left $\mathcal{O}$-module and

$\Lambda \otimes \mathbb{Q}$ is a left B-module of the same rank over $\mathbb{Q}$ as B. Since every left B-module is free (cf. [33], Chapter 9), there is an element $v_0 \in V$ such that $\Lambda \otimes \mathbb{Q} = B \cdot v_0$ and therefore $\Lambda = \mathcal{I} \cdot v_0$ for some left $\mathcal{O}$-ideal $\mathcal{I} \subset B$.

Let $\mathrm{Pic}_\ell(\mathcal{O})$ be the pointed set of left (projective) ideals of $\mathcal{O}$ up to principal ideals. By a theorem of Eichler ([5], [6]), the reduced norm on B induces a bijection of sets $\mathrm{n} : \mathrm{Pic}_\ell(\mathcal{O}) \xrightarrow{\sim} \mathrm{Pic}(F)$ onto the class group of F.

Note that the left ideal $\mathcal{I}$ is determined by A up to principal ideals and we can choose (and fix) a representative of $\mathcal{I}$ in its class in $\mathrm{Pic}_\ell(\mathcal{O})$ such that $\mathrm{n}(\mathcal{I}) \subset F$ is coprime with $\mathcal{D}_\mathcal{O}$. This is indeed possible because B is totally indefinite: it is a consequence of the Hasse-Schilling-Maass Norm Theorem, Eichler's Theorem quoted above and the natural epimorphism of ray class groups $\mathrm{Cl}^{\mathcal{D}_\mathcal{O}}(F) \to \mathrm{Cl}(F)$ of ideals of F ([23], Chapter VI, Section 6).

Let $\rho_a : B \hookrightarrow \mathrm{End}(V) \simeq \mathrm{M}_{2n}(\mathbb{C})$ and $\rho_r : \mathcal{O} \hookrightarrow \mathrm{End}(\Lambda) \simeq \mathrm{M}_{4n}(\mathbb{Z})$ denote the analytic and rational representations of B and $\mathcal{O}$ on V and Λ, respectively. It is well known that $\rho_r \sim \rho_a \bigoplus \bar{\rho}_a$ and it follows that, in an appropriate basis,

$$\rho_a(\beta) = \mathrm{diag}(\eta_{\sigma_i}(\beta))$$

for any $\beta \in B$ (cf. [18], Chapter 9, Lemma 1.1). Moreover, this basis can be chosen so that the coordinates of v_0 are $(\tau_1, 1, ..., \tau_n, 1)$ for certain $\tau_i \in \mathbb{C}$, $\mathrm{Im}(\tau_i) \neq 0$. The choice of the element v_0 fixes an isomorphism of real vector spaces $B \otimes \mathbb{R} \simeq V$.

Conversely, for any choice of a vector $v_0 = (\tau_1, 1, ..., \tau_n, 1) \in V$ with $\mathrm{Im}(\tau_i) \neq 0$ and a left $\mathcal{O}$-ideal $\mathcal{I}$ in B, we can consider the complex torus V/Λ with $\Lambda = \mathcal{I} \cdot v_0$ and B acting on V via the fixed diagonal analytic representation ρ_a. The torus V/Λ admits a polarization and can be embedded in a projective space. In consequence, it is the set of complex points of an abelian variety A such that $\mathrm{End}(A) \supseteq \mathcal{O}$ (cf. [29] and [18], Chapter 9, Section 4).

If $\mathcal{O} = \mathcal{O}_\ell(\mathcal{I}) = \{\beta \in B, \beta\mathcal{I} \subseteq \mathcal{I}\}$ is the left order of $\mathcal{I}$ in B, it holds that for the choice of v_0 in a dense subset of V, we exactly have $\mathrm{End}(A) = \mathcal{O}$. Besides, for v_0 in a subset of measure zero of V, A fails to be simple and it is isogenous to the square A_0^2 of an abelian variety of dimension n such that $\mathrm{End}(A_0)$ is an order in a purely imaginary quadratic extension of F ([30], Section 9.4).

Let $\mathrm{NS}(A) = \mathrm{Pic}(A)/\mathrm{Pic}^0(A)$ be the Néron-Severi group of line bundles on A up to algebraic equivalence. Two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2 \in \mathrm{NS}(A)$ are said to be isomorphic, denoted $\mathcal{L}_1 \simeq \mathcal{L}_2$, if there is an automorphism $\alpha \in \mathrm{Aut}(A)$ such that $\mathcal{L}_1 = \alpha^*(\mathcal{L}_2)$.

For an $\mathcal{O}$-left ideal J, let

$$J^\sharp = \{\beta \in B : \mathrm{tr}_{B/\mathbb{Q}}(J\beta) \subseteq \mathbb{Z}\}$$

be the codifferent of J over $\mathbb{Z}$. It is a right ideal of $\mathcal{O}$ projective over R_F. If we let $B_0 = \{\beta \in B : \mathrm{tr}(\beta) = 0\}$ denote the additive subgroup of pure quaternions of B,

we put $J_0^\sharp = J^\sharp \cap B_0$. Finally, we define $\mathcal{N}(J) = \mathrm{n}(J)\mathcal{O} = J\bar{J}$ to be the two-sided ideal of $\mathcal{O}$ generated by the ideal $\mathrm{n}(J)$ of F of reduced norms of elements in J.

The following theorem describes $\mathrm{NS}(A)$ intrinsically in terms of the arithmetic of B. In addition, it establishes when two line bundles on A are isomorphic and translates this into a certain conjugation relation in B. We keep the notations as above.

Theorem 2.2. *There is a natural isomorphism*

$$\begin{array}{ccc} \mathrm{NS}(A) & \xrightarrow{\sim} & \mathcal{N}(\mathcal{I})_0^\sharp \\ \mathcal{L} & \mapsto & \mu = \mu(\mathcal{L}) \end{array}$$

between the Néron-Severi group of A and the group of pure quaternions of the codifferent of the two-sided ideal $\mathcal{N}(\mathcal{I})$. Moreover, for any two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2 \in \mathrm{NS}(A)$, we have that $\mathcal{L}_1 \simeq \mathcal{L}_2$ if and only if there exists $\alpha \in \mathcal{O}^*$ such that $\mu(\mathcal{L}_1) = \bar{\alpha}\mu(\mathcal{L}_2)\alpha$.

Proof. By the Appell-Humbert Theorem, the first Chern class allows us to interpret a line bundle $\mathcal{L} \in \mathrm{NS}(A)$ as a Riemann form: an $\mathbb{R}$-alternate bilinear form $E = c_1(\mathcal{L}) : V \times V \to \mathbb{R}$ such that $E(\Lambda \times \Lambda) \subset \mathbb{Z}$ and $E(\sqrt{-1}u, \sqrt{-1}v) = E(u, v)$ for all u, $v \in V$. Fix a line bundle $\mathcal{L}$ on A and let $E = c_1(\mathcal{L})$ be the corresponding Riemann form. The linear map $B \to \mathbb{Q}$, $\beta \mapsto E(\beta v_0, v_0)$ is a trace form on B and hence, by the nondegeneracy of $\mathrm{tr}_{B/\mathbb{Q}}$, there is a unique element $\mu \in B$ such that $E(\beta v_0, v_0) = \mathrm{tr}_{B/\mathbb{Q}}(\mu\beta)$ for any $\beta \in B$. Since E is alternate, $E(av_0, av_0) = \mathrm{tr}_{F/\mathbb{Q}}(a^2\mathrm{tr}_{B/F}(\mu)) = 0$ for any $a \in F$. It follows again from the nondegeneracy of $\mathrm{tr}_{F/\mathbb{Q}}$ and the fact that the squares F^{*2} span F as a $\mathbb{Q}$-vector space that $\mathrm{tr}_{B/F}(\mu) = 0$. Thus $\mu^2 + \delta = 0$ for some $\delta \in F$.

The line bundle $\mathcal{L}$ induces an anti-involution ϱ on B called the Rosati involution. It is characterized by the rule $E(u, \beta v) = E(\beta^\varrho u, v)$ for any $\beta \in B$ and u, $v \in V$. From the discussion above, it must be $\beta^\varrho = \mu^{-1}\bar{\beta}\mu$ and we conclude that the Riemann form $E = c_1(\mathcal{L})$ attached to the line bundle $\mathcal{L}$ on A is

$$\begin{array}{rccc} E := E_\mu : & V \times V & \longrightarrow & \mathbb{R} \\ & (u, v) & \mapsto & \mathrm{tr}_{B\otimes_{\mathbb{Q}}\mathbb{R}/\mathbb{R}}(\mu\gamma\bar{\beta}) \end{array}$$

where $\mu \in B$ is determined as above and γ and β are elements in $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R})^n$ such that $u = \gamma v_0$ and $v = \beta v_0$. Since $E(\Lambda \times \Lambda) \subset \mathbb{Z}$ and $\mathrm{tr}(\mu) = 0$, we deduce that $\mu \in \mathcal{N}(\mathcal{I})_0^\sharp$.

Conversely, one checks that any element $\mu \in \mathcal{N}(\mathcal{I})_0^\sharp$ defines a Riemann form E_μ which is in turn the first Chern class of a line bundle $\mathcal{L}$ on A. Indeed, since $\mu \in \mathcal{N}(\mathcal{I})^\sharp$, the form E_μ is integral over the lattice $\Lambda = \mathcal{I} \cdot v_0$ and E_μ is alternate because $\mathrm{tr}(\mu) = 0$. Moreover, let $\iota = \mathrm{diag}(\iota_1, ..., \iota_n) \in \mathrm{GL}_{2n}(\mathbb{R})$ with $\iota_i \in \mathrm{GL}_2(\mathbb{R})$ and $\iota_i^2 + 1 = 0$, be a matrix such that $\iota \cdot v_0 = \sqrt{-1}v_0$. Then $E_\mu(\sqrt{-1}u, \sqrt{-1}v) =$

$E_\mu(\gamma\sqrt{-1}v_0, \beta\sqrt{-1}v_0) = \mathrm{tr}(\mu\gamma\iota\bar{\iota}\bar{\beta}) = E_\mu(u,v)$ for all u, $v \in V$. This concludes the proof of the first part of the theorem.

As for the second, we note that the first Chern class of the pull-back $\alpha^*\mathcal{L}$ of a line bundle $\mathcal{L}$ on A by an automorphism $\alpha \in \mathrm{Aut}(A) = \mathcal{O}^*$ is represented by the Riemann form $\alpha^*E : V \times V \to \mathbb{R}$ defined by $(u,v) \mapsto E(\alpha u, \alpha v)$, where $E = c_1(\mathcal{L})$ is the Riemann form associated to $\mathcal{L}$. Hence, if $\mathcal{L}_1 = \alpha^*(\mathcal{L}_2)$, then $\mathrm{tr}(\mu_1\gamma\bar{\beta}) = \mathrm{tr}(\mu_2\alpha\gamma\bar{\beta}\bar{\alpha}) = \mathrm{tr}(\bar{\alpha}\mu_2\alpha\gamma\bar{\beta})$ for all γ and $\beta \in B$ and this is satisfied if and only if $\mu_1 = \bar{\alpha}\mu_2\alpha$, by the nondegeneracy of the trace form. Reciprocally, one checks that if $\mu_1 = \bar{\alpha}\mu_2\alpha$ for some $\alpha \in \mathcal{O}^*$, then $E_{\mu_1} = \alpha^*E_{\mu_2}$ and therefore $\mathcal{L}_1 = \alpha^*\mathcal{L}_2$. $\square$

In view of Theorem 2.2, we identify the first Chern class $c_1(\mathcal{L})$ of a line bundle $\mathcal{L}$ on A with the quaternion $\mu = \mu(\mathcal{L}) \in B_0$ such that E_μ is the Riemann form associated to $\mathcal{L}$. We are led to introduce the following equivalence relation, that was studied (over B) by O'Connor and Pall in the 1930s and Pollack in the 1960s (cf. [25]).

Definition 2.3. Two quaternions μ_1 and $\mu_2 \in B$ are *Pollack conjugated* over $\mathcal{O}$ if $\mu_1 = \bar{\alpha}\mu_2\alpha$ for some unit $\alpha \in \mathcal{O}^*$. We will denote it by $\mu_1 \sim_p \mu_2$.

3. Isomorphism classes of line bundles and Eichler theory on optimal embeddings

As in the previous section, let A denote an abelian variety with quaternionic multiplication by an hereditary order $\mathcal{O}$ in a totally indefinite division quaternion algebra B over a totally real number field F. As is well-known, a line bundle $\mathcal{L} \in \mathrm{NS}(A)$ induces a morphism $\varphi_{\mathcal{L}} : A \to \hat{A}$ defined by $P \mapsto \mathrm{t}_P^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$, where $t_P : A \to A$ denotes the translation by P map. Since A is simple, any nontrivial line bundle $L \in \mathrm{NS}(A)$ is nondegenerate: $\varphi_{\mathcal{L}}$ is an isogeny with finite kernel $K(\mathcal{L})$. We say that $\mathcal{L}$ is *principal* if $K(\mathcal{L})$ is trivial, that is, if $\varphi_{\mathcal{L}} : A \to \hat{A}$ is an isomorphism.

Proposition 3.1. *Let $\mathcal{L}$ be a line bundle on A and let $c_1(\mathcal{L}) = \mu$ be its first Chern class for some element $\mu \in B$ such that $\mu^2 + \delta = 0$ and $\delta \in F$. Then*

$$\deg(\varphi_{\mathcal{L}}) = \mathrm{N}_{F/\mathbb{Q}}(\vartheta_{F/\mathbb{Q}}^2 \cdot \mathrm{n}(\mathcal{I})^2 \cdot \mathcal{D}_{\mathcal{O}} \cdot \delta)^2$$

where $\vartheta_{F/\mathbb{Q}} = (R_F^\sharp)^{-1}$ is the different of F over $\mathbb{Q}$.

Proof. The degree $|K(\mathcal{L})|$ of $\varphi_{\mathcal{L}}$ can be computed in terms of the Riemann form as follows: $\deg(\varphi_{\mathcal{L}}) = \det(E_\mu(x_i, x_j)) = \det(\mathrm{tr}_{B/\mathbb{Q}}(\mu\beta_i\bar{\beta}_j))$, where $x_i = \beta_i v_0$ runs through a $\mathbb{Z}$-basis of the lattice Λ. We have $\det(\mathrm{tr}_{B/\mathbb{Q}}(\mu\beta_i\bar{\beta}_j)) = \mathrm{n}_{B/\mathbb{Q}}(\mu)^2 \det(\mathrm{tr}_{B/\mathbb{Q}}(\beta_i \cdot \bar{\beta}_j)) = \mathrm{n}_{B/\mathbb{Q}}(\mu)^2 \mathrm{disc}_{B/\mathbb{Q}}(\mathcal{I})^2 = (\mathrm{N}_{F/\mathbb{Q}}(\delta \cdot \mathrm{n}(\mathcal{I})^2 \cdot \vartheta_{F/\mathbb{Q}}^2 \cdot \mathcal{D}_{\mathcal{O}}))^2$. $\square$

As a consequence, we obtain the following criterion that establishes whether the abelian variety A admits a principal line bundle in terms of the arithmetic of the hereditary order $\mathcal{O} = \mathrm{End}(A)$ in B and the left ideal $\mathcal{I} \cong H_1(A, \mathbb{Z})$. Crucial in the proof of the theorem below is the classical theory of Eichler optimal embeddings.

For the sake of simplicity and unless otherwise stated, we assume for the rest of the article that $\vartheta_{F/\mathbb{Q}}$ and $\mathcal{D}_{\mathcal{O}}$ are coprime ideals in F. The general case can be dealt by means of the remark below.

Theorem 3.2. *The abelian variety A admits a* principal *line bundle if and only if the ideals $\mathcal{D}_{\mathcal{O}}$ and $\vartheta_{F/\mathbb{Q}} \cdot \mathrm{n}(\mathcal{I})$ of F are* principal.

Proof. Let $\mathcal{L}$ be a principal line bundle on A and let $E_\mu = c_1(\mathcal{L})$ be the associated Riemann form for some $\mu \in \mathcal{N}(\mathcal{I})_0^\sharp$ such that $\mu^2 + \delta = 0$. Since $\mathcal{L}$ is principal, the induced Rosati involution ϱ on $\mathrm{End}(A) \otimes \mathbb{Q} = B$ must also restrict to $\mathrm{End}(A) = \mathcal{O}$ and we already observed that $\beta^\varrho = \mu^{-1}\bar{\beta}\mu$. Therefore μ belongs to the normaliser group $\mathrm{Norm}_{B^*}(\mathcal{O})$ of $\mathcal{O}$ in B. The quotient $\mathrm{Norm}_{B^*}(\mathcal{O})/\mathcal{O}^*F^* \cong W$ is a finite abelian 2-torsion group and representatives w of W in $\mathcal{O}$ can be chosen so that the reduced norms $\mathrm{n}(w) \in R_F$ are only divisible by the prime ideals $\wp | \mathcal{D}_{\mathcal{O}}$ (cf. [31], p. 39, 99, [2]). Hence, we can express $\mu = u \cdot t \cdot w^{-1}$ for some $u \in \mathcal{O}^*$, $t \in F^*$ and $w \in W$.

Recall that $(\mathrm{n}(\mathcal{I}), \mathcal{D}_{\mathcal{O}}) = 1$ and $(\vartheta_{F/\mathbb{Q}}, \mathcal{D}_{\mathcal{O}}) = 1$. Since, from Proposition 3.1, $\mathrm{n}(\mathcal{I})^2 \cdot \vartheta_{F/\mathbb{Q}}^2 \cdot \mathcal{D}_{\mathcal{O}} = (\delta^{-1}) = (t^{-2} \cdot \mathrm{n}(w))$, we conclude that $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}} = (t^{-1})$ and $\mathcal{D}_{\mathcal{O}} = (\mathrm{n}(w))$ are principal ideals.

Conversely, suppose that $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}} = (t^{-1})$ and $\mathcal{D}_{\mathcal{O}} = (D)$ are principal ideals, generated by some elements t and $D \in F^*$. Let S be the ring of integers in $L = F(\sqrt{-D})$. Since any prime ideal $\wp | D$ ramifies in L, Eichler's theory of optimal embeddings guarantees the existence of an embedding $\iota : S \hookrightarrow \mathcal{O}$ of S into the quaternion order $\mathcal{O}$ (cf. [31], p. 45). Let $w = \iota(\sqrt{-D}) \in \mathcal{O}$ and let $\mu = t \cdot w^{-1}$. As one checks locally, $\mu \in \mathrm{Norm}_{B^*}(\mathcal{O}) \cap \mathcal{N}(\mathcal{I})_0^\sharp$ and, by Theorem 2.2 and Proposition 3.1, μ is the first Chern class of a principal line bundle on A. □

Corollary 3.3. *If $\mathcal{D}_{\mathcal{O}}$ and $\vartheta_{F/\mathbb{Q}} \cdot \mathrm{n}(\mathcal{I})$ are principal ideals, then A is self-dual, that is, $A \cong \hat{A}$.*

Remark 3.4. The case when $\vartheta_{F/\mathbb{Q}}$ and $\mathcal{D}_{\mathcal{O}}$ are non necessarily coprime is reformulated as follows: A admits a principal line bundle if and only if there is an integral ideal $\mathfrak{a} = \wp_1^{e_1} \cdot \ldots \cdot \wp_{2r}^{e_{2r}} | \vartheta_{F/\mathbb{Q}}$ in F such that both $\mathcal{D}_{\mathcal{O}} \cdot \mathfrak{a}^2$ and $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}} \cdot \mathfrak{a}^{-1}$ are principal ideals. In this case, A is also self-dual. The proof is *mutatis mutandi* the one given above.

Definition 3.5. The set of isomorphism classes of principal line bundles on A is

$$\Pi(A) = \{\mathcal{L} \in \mathrm{NS}(A) : \deg(\mathcal{L}) = 1\}/_{\simeq}.$$

Definition 3.6. Assume that $\mathcal{D}_{\mathcal{O}}$ is a principal ideal of F. Then, we let

$$\mathcal{P}(\mathcal{O}) = \{\mu \in \mathcal{O} : \mathrm{tr}(\mu) = 0, \mathrm{n}(\mu)R_F = \mathcal{D}_{\mathcal{O}}\}$$

and we define $P(\mathcal{O}) = \mathcal{P}(\mathcal{O})/_{\sim_p}$ to be the corresponding set of Pollack conjugation classes.

The above proof, together with Theorem 2.2, yields

Corollary 3.7. *Let A be an abelian variety with quaternionic multiplication by a maximal order $\mathcal{O}$. If $\mathcal{D}_{\mathcal{O}} = (D)$ and $\vartheta_{F/\mathbb{Q}} \cdot \mathrm{n}(\mathcal{I}) = (t^{-1})$ are principal ideals, the assignation*

$$\mathcal{L} \mapsto t \cdot c_1(\mathcal{L})^{-1}$$

induces a bijection of sets between $\Pi(A)$ and $P(\mathcal{O})$.

In view of Corollary 3.7, it is our aim to compute the cardinality $\pi(A) = |\Pi(A)|$ of the set of isomorphism classes of principal line bundles on an abelian variety A with quaternionic multiplication by a maximal order $\mathcal{O}$. Theorem 3.8 below exhibits a close relation between $\pi(A)$ and the class number of F and of certain orders in quadratic extensions L/F that embed in B.

Assume that there is a principal line bundle on A. Otherwise $\pi(A) = 0$ and there is nothing to compute. We assume also that $(\vartheta_{F/\mathbb{Q}}, \mathcal{D}_{\mathcal{O}}) = 1$. By Theorem 3.2, we know that $\mathcal{D}_{\mathcal{O}} = (D)$ for some $D \in F^*$. We have

Theorem 3.8. *Let A be an abelian variety with quaternionic multiplication by a maximal order $\mathcal{O}$. Then*

$$\pi(A) = \frac{1}{2h(F)} \sum_u \sum_S 2^{e_S} h(S),$$

where $u \in R_F^*/R_F^{*2}$ runs through a set of representatives of units in R_F up to squares and S runs through the (finite) set of orders in $F(\sqrt{-uD})$ such that $R_F[\sqrt{-uD}] \subseteq S$. Here, $2^{e_S} = |R_F^*/N_{F(\sqrt{-uD})/F}(S^*)|$.

The proof of Theorem 3.8 will be completed in Section 4. There are several remarks to be made for the sake of its practical applications.

Remark 3.9. Note that $R_F^{*2} \subseteq N_{L/F}(S^*)$ and hence $R_F^*/N_{L/F}(S^*)$ is naturally an $\mathbb{F}_2$-vector space. By Dirichlet's Unit Theorem, $e_S \leq [F : \mathbb{Q}] = n$. The case $F = \mathbb{Q}$ is trivial since $\mathbb{Z}^*/\mathbb{Z}^{*2} = \{\pm 1\}$. In the case of real quadratic fields F, explicit fundamental units $u \in R_F^*$ such that $R_F^*/R_F^{*2} = \{\pm 1, \pm u\}$ are well known. For totally real number fields of higher degree there is abundant literature on systems of units. See [17] and [32], Chapter 8 for an account.

Remark 3.10. Let $2R_F = \mathfrak{q}_1^{e_1} \cdot ... \cdot \mathfrak{q}_m^{e_m}$ be the decomposition of 2 into prime ideals in F. Fix a unit $u \in R_F^*$. Then, the conductor $\mathfrak{f}$ of $R_F[\sqrt{-uD}]$ in L over R_F is $\mathfrak{f} = \prod_{\substack{\mathfrak{q}|2 \\ \mathfrak{q}\nmid D}} \mathfrak{q}^{a_{\mathfrak{q}}}$, for some $0 \leq a_{\mathfrak{q}} \leq e_{\mathfrak{q}}$.

Further, the conductor $\mathfrak{f}$ can be completely determined in many cases as follows. For a prime ideal $\mathfrak{q}|2$ such that $\mathfrak{q} \nmid D$, let π be a local uniformizer of the completion of F at $\mathfrak{q}$ and $k = \mathbb{F}_{2^f}$ be the residue field. Let $e = e_{\mathfrak{q}} \geq 1$. Since $-uD \in R_{F_{\mathfrak{q}}}^*$, we have $-uD = x_0 + x_k\pi^k + x_{k+1}\pi^{k+1} + ...$ for some $1 \leq k \leq \infty$ and x_i in a system of

representatives of $\mathbb{F}_{2^f}$ in $R_{F_\mathfrak{q}}$ such that $\bar{x_0}$ and $\bar{x_k} \neq 0$. Here we agree to set $k = \infty$ if $-uD = x_0$. Then, $\min([\frac{k}{2}], e) \leq a_\mathfrak{q} \leq e$ and we exactly have

$$a_\mathfrak{q} = \begin{cases} [\frac{k}{2}] & \text{if } k \leq e+1, \\ e & \text{if } [\frac{k}{2}] \geq e. \end{cases}$$

Otherwise, if $[\frac{k}{2}] < e < k-1$, the determination of $a_\mathfrak{q}$ depends on the choice of the system of representatives of $\mathbb{F}_{2^f}$ in $R_{F_\mathfrak{q}}$ and it deserves a closer inspection.

This gives an easy criterion for deciding whether $R_F[\sqrt{-uD}]$ is the ring of integers of $F(\sqrt{-uD})$ (that is, $\mathfrak{f} = 1$). In any case, the set of orders S in $L = F(\sqrt{-uD})$ that contain $R_F[\sqrt{-uD}]$ can be described as follows. Any order $S \supseteq R_F[\sqrt{-uD}]$ has conductor $\mathfrak{f}_S | \mathfrak{f}$ and for every ideal $\mathfrak{f}' | \mathfrak{f}$ there is a unique order $S \supseteq R_F[\sqrt{-uD}]$ of conductor $\mathfrak{f}'$. Further, $\mathfrak{f}_S | \mathfrak{f}_T$ if and only if $S \supseteq T$. We omit the details of the proof of these facts.

In order to prove Theorem 3.8, we begin by an equivalent formulation of it. As it was pointed out in Corollary 3.7, the first Chern class induces a bijection of sets between $\Pi(A)$ and the set of Pollack conjugation classes $P(\mathcal{O})$. For $u \in R_F^*$, let us write $\mathcal{P}(u, \mathcal{O}) := \{\mu \in \mathcal{O} : \mu^2 + uD = 0\}$. Observe that $\mathcal{P}(\mathcal{O})$ is the disjoint union of the sets $\mathcal{P}(u_k, \mathcal{O})$ as u_k run among units in any set of representatives of R_F^*/R_F^{*2}.

Any quaternion $\mu \in \mathcal{P}(u, \mathcal{O})$ induces an embedding

$$\begin{array}{rcl} i_\mu : F(\sqrt{-uD}) & \hookrightarrow & B \\ a + b\sqrt{-uD} & \mapsto & a + b\mu \end{array}$$

for which $i_\mu(R_F[\sqrt{-uD}]) \subset \mathcal{O}$. The following definition is due to Eichler.

Definition 3.11. Let S be an order over R_F in a quadratic algebra L over F. An embedding $i : S \hookrightarrow \mathcal{O}$ is *optimal* if $i(S) = i(L) \cap \mathcal{O}$.

For any $\mu \in \mathcal{P}(u, \mathcal{O})$ there is a uniquely determined order $S_\mu \supseteq R_F[\sqrt{-uD}]$ such that i_μ is optimal at S_μ. Moreover, two equivalent quaternions $\mu_1 \sim_p \mu_2 \in \mathcal{P}(u, \mathcal{O})$ are optimal at the same order S. Indeed, if $\alpha \in \mathcal{O}^*$ is such that $\mu_1 = \bar{\alpha}\mu_2\alpha$, then α is forced to have reduced norm $\mathrm{n}(\alpha) = \pm 1$. Hence $\bar{\alpha} = \pm\alpha^{-1} \in \mathcal{O}^*$ and the observation follows since α normalizes $\mathcal{O}$. Conversely, any optimal embedding $i : S \hookrightarrow \mathcal{O}$, $S \supseteq R_F[\sqrt{-uD}]$ determines a quaternion $\mu = i(\sqrt{-uD}) \in \mathcal{P}(u, \mathcal{O})$.

For any quadratic order S over R_F, let $\mathcal{P}(S, \mathcal{O})$ denote the set of optimal embeddings of S in $\mathcal{O}$ and $P(S, \mathcal{O}) = \mathcal{P}(S, \mathcal{O})/_{\sim_p}$. We obtain a natural identification of sets

$$P(\mathcal{O}) = \sqcup_k \sqcup_S P(S, \mathcal{O}),$$

where S runs through the set of quadratic orders $S \supseteq R_F[\sqrt{-u_k D}]$ for any unit u_k in a set of representatives of R_F^*/R_F^{*2}. Hence, in order to prove Theorem 3.8, it is

enough to show that, for any quadratic order $S \supseteq R_F[\sqrt{-uD}]$, $u \in R_F^*$, it holds that

$$p(S,\mathcal{O}) := |P(S,\mathcal{O})| = \frac{2^{e_S-1}h(S)}{h(F)}.$$

Since this question is interesting on its own, we will make our statements in greater generality in the next section.

4. **Pollack conjugation versus Eichler conjugation**

Let F be a number field and let B be a division quaternion algebra over F. In [25], Pollack studied the obstruction for two pure quaternions μ_1 and $\mu_2 \in B$ with the same reduced norm to be conjugated over B^*, that is, $\mu_1 = \bar{\alpha}\mu_2\alpha$ with $\alpha \in B^*$, and expressed it in terms of the 2-torsion subgroup $\mathrm{Br}_2(F)$ of the Brauer group $\mathrm{Br}(F)$ of F. He further investigated the solvability of the equation $\mu_1 = \bar{\alpha}\mu_2\alpha$ over $\mathcal{O}^*$ for quaternions μ_1 and μ_2 in a maximal order $\mathcal{O}$ of B.

As a refinement of his considerations, it is natural to consider the set of orbits of pure quaternions $\mu \in \mathcal{O}$ of fixed reduced norm $\mathrm{n}(\mu) = d \in F^*$ under the action of the group of units $\mathcal{O}^*$ by Pollack conjugation. As already mentioned, a necessary condition for $\mu_1 \sim_p \mu_2$ over $\mathcal{O}^*$ is that μ_1 and μ_2 induce an optimal embedding at the same quadratic order $F(\sqrt{-d}) \supset S \supseteq R_F[\sqrt{-d}]$. We will drop the restriction on $\mathcal{O}$ to be maximal in our statements.

The connection between optimal embeddings orbits and class numbers is made possible by the theory of Eichler. However, in contrast to Pollack conjugation, two optimal embeddings $i, j : S \hookrightarrow \mathcal{O}$ lie on the same conjugation class in the sense of Eichler, written $i \sim_e j$, if there exists $\alpha \in \mathcal{O}^*$ such that $i = \alpha^{-1}j\alpha$. We let $E(S,\mathcal{O}) = \mathcal{P}(S,\mathcal{O})/_{\sim_e}$ be the set of Eichler conjugation classes of optimal embeddings of S into an order $\mathcal{O}$ of B and denote $e(S,\mathcal{O}) = |E(S,\mathcal{O})|$.

Proposition 4.1. *Let S be an order in a quadratic algebra L over F and let $\mathcal{O}$ be an order in a division quaternion algebra B over F. Then, the number of Pollack conjugation classes is*

$$p(S,\mathcal{O}) = |\mathrm{n}(\mathcal{O}^*)/N_{L/F}(S^*)| \cdot \frac{e(S,\mathcal{O})}{2}.$$

Proof. Let us agree to say that two pure quaternions μ_1 and $\mu_2 \in \mathcal{O}$ lie in the same $\pm$Eichler conjugation class if there exists $\alpha \in \mathcal{O}^*$ such that $\mu_1 = \pm\alpha^{-1}\mu_2\alpha$. We shall denote it by $\mu_1 \sim_{\pm e} \mu_2$ and $E_\pm(S,\mathcal{O}) = \mathcal{P}(S,\mathcal{O})/_{\sim_{\pm e}}$. The identity map $\mu \mapsto \mu$ descends to a natural surjective map

$$\rho : P(S,\mathcal{O}) \quad \to \quad E_\pm(S,\mathcal{O})$$

and the proposition follows from the following lemma.

Lemma 4.2. *Let* $e_S = \dim_{\mathbb{F}_2}(\mathrm{n}(\mathcal{O}^*)/N_{L/F}(S^*))$. *Let* $\mu \in \mathcal{P}(S,\mathcal{O})$ *and let* $\varepsilon_\mu = 1$ *if* $\mu \sim_e -\mu$ *and* $\varepsilon_\mu = 2$ *otherwise. Then, in the* $\pm$*Eichler conjugation class* $\{\pm\alpha^{-1}\mu\alpha : \alpha \in \mathcal{O}^*\}$ *of* μ*, there are exactly* $\varepsilon_\mu 2^{e_S-1}$ *Pollack conjugation classes of pure quaternions.*

Proof of Lemma 4.2. Suppose first that $\varepsilon_\mu = 1$. Then, the $\pm$Eichler conjugation class of $\mu \in \mathcal{P}(S,\mathcal{O})$ is $\{\alpha^{-1}\mu\alpha : \alpha \in \mathcal{O}^*\}$. Let $\gamma \in \mathcal{O}^*$ be such that $-\mu = \gamma\mu\gamma^{-1} = \gamma^{-1}\mu\gamma$. We claim that, for any given $\alpha \in \mathcal{O}^*$, it holds that $\mu \sim_p \alpha^{-1}\mu\alpha$ if and only if $\mathrm{n}(\alpha) \in N_{L/F}(S^*) \cup (-\mathrm{n}(\gamma)N_{L/F}(S^*))$. Indeed, if $\mu \sim_p \alpha^{-1}\mu\alpha$, let $\beta \in \mathcal{O}^*$ with $\mathrm{n}(\beta) = \pm 1$ be such that $\bar{\beta}\alpha^{-1}\mu\alpha\beta = \mu$. If $\mathrm{n}(\beta) = 1$, then $\alpha\beta\mu = \mu\alpha\beta$ and hence $\alpha\beta \in L \cap \mathcal{O}^* = S^*$. Thus $\mathrm{n}(\alpha\beta) = \mathrm{n}(\alpha) \in N_{L/F}(S^*)$. If $\mathrm{n}(\beta) = -1$, a similar argument shows that $\mathrm{n}(\alpha) \in -\mathrm{n}(\gamma) \cdot N_{L/F}(S^*)$. Conversely, let $\mathrm{n}(\alpha) = v \in N_{L/F}(S^*) \cup (-\mathrm{n}(\gamma)N_{L/F}(S^*))$ and let $s \in S^*$ be such that $N_{L/F}(s) = v$ or $-v\mathrm{n}(\gamma)^{-1}$. Since μ induces an embedding $S \hookrightarrow \mathcal{O}$, we can regard s as an element in $\mathcal{O}^*$ such that $\mathrm{n}(s) = v$ or $-v\mathrm{n}(\gamma)^{-1}$ and $s\mu = \mu s$. Hence $\alpha^{-1}\mu\alpha = \alpha^{-1}s\mu s^{-1}\alpha = (\alpha^{-1}s)\mu\overline{(\alpha^{-1}s)}$ or $(\alpha^{-1}s\gamma^{-1})\mu\overline{(\alpha^{-1}s\gamma^{-1})}$. This proves the claim.

Since B is division, Pollack's Theorem on Pall's Conjecture ([25], Theorem 4) applies to show that $-\mathrm{n}(\gamma) \notin N_{L/F}(S^*)$. We then conclude that the distinct Pollack conjugation orbits in $\{\alpha^{-1}\mu\alpha : \alpha \in \mathcal{O}^*\}$ are exactly the classes

$$\mathcal{C}_u = \{\alpha^{-1}\mu\alpha : \mathrm{n}(\alpha) \in uN(S^*) \cup (-\mathrm{n}(\gamma)uN(S^*))\}$$

as $u \in \mathrm{n}(\mathcal{O}^*)$ runs through a set of representatives in $\mathrm{n}(\mathcal{O}^*)/\langle -\mathrm{n}(\gamma), N(S^*)\rangle$. There are 2^{e_S-1} of them.

Assume that $\varepsilon_\mu = 2$, that is, $\mu \not\sim_e -\mu$. Then, the $\pm$Eichler conjugation class of $\mu \in \mathcal{P}(S,\mathcal{O})$ is $\{\alpha^{-1}\mu\alpha : \alpha \in \mathcal{O}^*\} \cup \{-\alpha^{-1}\mu\alpha : \alpha \in \mathcal{O}^*\}$. As in the previous case, it is shown that $\mu \sim_p \alpha^{-1}\mu\alpha$ if and only if $\mathrm{n}(\alpha) \in N_{L/F}(S^*)$ and $\mu \sim_p -\alpha^{-1}\mu\alpha$ if and only if $\mathrm{n}(\alpha) \in -N_{L/F}(S^*)$. We obtain that, as $u \in \mathrm{n}(\mathcal{O}^*)$ runs through a set of representatives in $\mathrm{n}(\mathcal{O}^*)/N(S^*)$, the 2^{e_S} distinct Pollack conjugation classes in the $\pm$Eichler conjugation class of the quaternion $\mu \in \mathcal{P}(S,\mathcal{O})$ are

$$\mathcal{C}'_u = \{\alpha^{-1}\mu\alpha : \mathrm{n}(\alpha) \in uN(S^*)\} \cup \{-\alpha^{-1}\mu\alpha : \mathrm{n}(\alpha) \in -uN(S^*)\}. \quad \square$$

Proof of Theorem 3.8. Firstly, under the assumptions of Theorem 3.8, the Hasse-Schilling-Maass Norm Theorem in its integral version ([31], p. 90) asserts that $\mathrm{n}(\mathcal{O}^*) = R_F^*$. Secondly, we have that

$$e(S,\mathcal{O}) = \frac{h(S)}{h(F)}$$

for any $S \supseteq R_F[\sqrt{-uD}]$, $u \in R_F^*$. This follows from a theorem of Eichler (cf. [31], p. 98) together with Remark 3.10. The combination of these facts together with the discussion at the end of the Section 3 and Proposition 4.1 yield the theorem. $\square$

Remark 4.3. In view of Proposition 4.1, the effective computation of the number of Pollack conjugation classes $p(S,\mathcal{O})$ for arbitrary orders lies on the computability of the groups $N_{L/F}(S^*)$ and $\mathrm{n}(\mathcal{O}^*)$ and the number $e(S,\mathcal{O})$. The study of the former depends on the knowledge of the group of units S^* and there is abundant literature on the subject. If $\mathcal{O}$ is an Eichler order, the Hasse-Schilling-Maass Norm Theorem in its integral version ([31], p. 90) describes $\mathrm{n}(\mathcal{O}^*)$ in terms of the archimedean ramified places of B. Finally, there are several manuscripts which deal with the computation of the numbers $e(S,\mathcal{O})$. See [31], p. 45, for Eichler orders and [11] and [2] for Gorenstein and Bass orders.

Since it will be of use later, we will consider in Proposition 4.4 below a stronger form of Eichler's Theorem. We keep the notations as in Theorem 3.8. Let $S \supseteq R_F[\sqrt{-uD}]$, $u \in R_F^*$, be a quadratic order and let H_S be the *ring class field* of S over $L = F(\sqrt{-uD})$. The Galois group $\mathrm{Gal}(H_S/L)$ is isomorphic, via the Artin reciprocity map, to the Picard group $\mathrm{Pic}(S)$ of classes of locally invertible ideals of S. In the particular case that S is the ring of integers of L, then H_S is the *Hilbert class field* of L. The quadratic extension L/F is ramified at the prime ideals $\wp|D$ and recall that, by Remark 3.10, these prime ideals do not divide the conductor of S. Therefore L and H_S are linearly disjoint over F, that is, $F = L \cap H_S$. The norm induces a map $N_{L/F} : \mathrm{Pic}(S) \to \mathrm{Pic}(R_F)$ that, by the reciprocity isomorphism can be interpreted as the restriction map $\mathrm{Gal}(H_S/L) \to \mathrm{Gal}(L \cdot H_F/L) \simeq \mathrm{Gal}(H_F/F)$ (cf. [23], Chapter VI, Section 5). In particular, we have an exact sequence

$$0 \to \Delta \to \mathrm{Pic}(S) \overset{N_{L/F}}{\to} \mathrm{Pic}(R_F) \to 0.$$

Here, $\Delta = \mathrm{Ker}\ (N_{L/F})$ can be viewed as the Galois group of H_S over the fixed field $L_\triangle$ of H_S by Δ. The group $\Delta = \mathrm{Gal}(H_S/L_\triangle)$ acts on $E(S,\mathcal{O})$ by a reciprocity law as follows: let $i : S \hookrightarrow \mathcal{O}$ be an optimal embedding and let $\tau \in \mathrm{Gal}(H_S/L_\triangle)$. Then let $\mathfrak{b} = [\tau, H_S/L]$ be the locally invertible ideal in S corresponding to τ by the Artin's reciprocity map. Since the reduced norm on B induces a bijection of sets $\mathrm{n} : \mathrm{Pic}_\ell(\mathcal{O}) \simeq \mathrm{Pic}(F)$ and $N_{L/K}(\mathfrak{b})$ is a principal ideal in F, it follows that $i(\mathfrak{b})\mathcal{O} = \beta\mathcal{O}$ is a principal right ideal of $\mathcal{O}$ and we can choose a generator $\beta \in \mathcal{O}$. Then τ acts on $i \in E(S,\mathcal{O})$ as

$$i^\tau = \beta^{-1} i \beta.$$

It can be checked that this action does not depend on the choice of the ideal $\mathfrak{b}$ in its class in $\mathrm{Pic}(S)$ nor on the choice of the element $\beta \in \mathcal{O}$. Moreover, a local argument shows that this action is free. Since $|\Delta| = |E(S,\mathcal{O})|$, we obtain

Proposition 4.4. *The action of Δ on the set of Eichler conjugacy classes of optimal embeddings $E(S,\mathcal{O})$ is free and transitive.*

The above action acquires a real arithmetic meaning and coincides with Shimura's reciprocity law in the particular case that L is a CM field over F. In this situation,

$E(S,\mathcal{O})$ can also be interpreted as the set of *Heegner points* on a Shimura variety $\mathfrak{X}$ on which the Galois group Δ is acting (cf. [30], Section 9.10).

5. The index of a nondegenerate line bundle

Let $\mathcal{L} \in \mathrm{NS}(A)$ be a nondegenerate line bundle on an abelian variety $A/\mathbb{C}$. By Mumford's Vanishing Theorem, there is a unique integer $i(\mathcal{L})$ such that $H^{i(\mathcal{L})}(A,\mathcal{L}) \neq 0$ and $H^j(A,\mathcal{L}) = 0$ for all $j \neq i(\mathcal{L})$ (cf. [21], Chapter III, Section 16, p. 150). The so-called index $i(\mathcal{L})$ does only depend on the class of $\mathcal{L}$ in $\mathrm{NS}(A)$ and we have $0 \leq i(\mathcal{L}) \leq g = \dim(A)$.

If H is the hermitian form associated to a line bundle $\mathcal{L}$, the index $i(\mathcal{L})$ agrees with the number of negative eigenvalues of H ([21], Chapter III, Section 16, p. 162). By the Riemann-Roch Theorem, $|K(\mathcal{L})| = |\mathrm{Ker}\ \varphi_{\mathcal{L}} : A \to \hat{A}| = \dim(H^{i(\mathcal{L})}(A,\mathcal{L}))$. In particular, $\mathcal{L}$ is *principal* when $\dim(H^{i(\mathcal{L})}(A,\mathcal{L})) = 1$. Finally, $\mathcal{L}$ is a polarization, i. e., an ample line bundle, if and only if $i(\mathcal{L}) = 0$.

Let A have quaternionic multiplication by an hereditary order $\mathcal{O}$ in B. It is our aim to compute the index of a line bundle $\mathcal{L}$ on A in terms of the quaternion $\mu = c_1(\mathcal{L})$. For these purposes, we introduce the following notation. Let $\mu \in B_0$ be a pure quaternion. It satisfies $\mu^2 + \delta = 0$ for some $\delta \in F^*$. For any immersion $\sigma : F \hookrightarrow \mathbb{R}$, let $\nu_\sigma = \nu_\sigma(\mu) \in \mathrm{GL}_2(\mathbb{R})$ be such that $\nu_\sigma \mu^\sigma \nu_\sigma^{-1} = \omega_\sigma$, where

$$\omega_\sigma = \begin{cases} \begin{pmatrix} 0 & \sqrt{\sigma(\delta)} \\ -\sqrt{\sigma(\delta)} & 0 \end{pmatrix} & \text{if } \sigma(\delta) > 0, \\ \begin{pmatrix} \sqrt{\sigma(-\delta)} & 0 \\ 0 & -\sqrt{\sigma(-\delta)} \end{pmatrix} & \text{otherwise.} \end{cases}$$

We say that μ has *positive* or *negative orientation at* σ according to the sign of the real number $\det(\nu_\sigma)$. Although ν_σ is not uniquely determined by μ, the sign $\mathrm{sgn}(\det(\nu_\sigma))$ is. Thus, to any pure quaternion $\mu \in B_0$, we can attach a signature

$$\mathrm{sgn}(\mu) = \mathrm{sgn}(\det(\nu_\sigma)) \in \{\pm 1\}^n.$$

Motivated by the following theorem, we say that a pure quaternion μ is *ample* with respect to A if it has the same orientation as $v_0 \in V = \mathrm{Lie}(A)$: $\mathrm{sgn}(\mu) = \mathrm{sgn}(\mathrm{Im}(\tau_i))$. For any real immersion $\sigma : F \hookrightarrow \mathbb{R}$, define the local archimedean index $i_\sigma(\mu)$ of μ by

$$i_\sigma(\mu) = \begin{cases} 0 & \text{if } \sigma(\delta) > 0 \text{ and } \det(\nu_\sigma) \cdot \mathrm{Im}(\tau_\sigma) > 0, \\ 1 & \text{if } \sigma(\delta) < 0, \\ 2 & \text{if } \sigma(\delta) > 0 \text{ and } \det(\nu_\sigma) \cdot \mathrm{Im}(\tau_\sigma) < 0. \end{cases}$$

With these notations we have

Theorem 5.1. *The index of* $\mathcal{L} \in \mathrm{NS}(A)$ *is*

$$i(\mathcal{L}) = \sum_{\sigma : F \hookrightarrow \mathbb{R}} i_\sigma(\mu).$$

Proof. The index of $i(\mathcal{L})$ coincides with the number of negative eigenvalues of the hermitian form H_μ associated to the line bundle $\mathcal{L}$. If we regard $\mathrm{M}_2(\mathbb{R}) \times \overset{n}{\ldots} \times \mathrm{M}_2(\mathbb{R})$ embedded diagonally in $\mathrm{M}_{2n}(\mathbb{R})$, there is an isomorphism of real vector spaces between $B \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathrm{M}_2(\mathbb{R}) \times \overset{n}{\ldots} \times \mathrm{M}_2(\mathbb{R})$ explicitly given by the map $\beta \mapsto \beta \cdot v_0$. The complex structure that $\mathrm{M}_2(\mathbb{R})^n$ inherits from that of V is such that $\{0\} \times \ldots \times \mathrm{M}_2(\mathbb{R}) \times \ldots \times \{0\}$ are complex vector subspaces of $\mathrm{M}_2(\mathbb{R})^n$ and we may choose a $\mathbb{C}$-basis of V of the form $\{\mathrm{diag}\ (\beta_1, 0, ..., 0) \cdot v_0, \mathrm{diag}\ (\gamma_1, 0, ..., 0) \cdot v_0, ..., \mathrm{diag}\ (0, ..., 0, \beta_n) \cdot v_0, \mathrm{diag}\ (0, ..., \gamma_n) \cdot v_0\}$ for β_i, $\gamma_i \in \mathrm{M}_2(\mathbb{R})$.

Let $\iota = \mathrm{diag}(\iota_\sigma) \in \mathrm{GL}_{2n}(\mathbb{R})$ be such that $\iota \cdot v_0 = \sqrt{-1} v_0$. For any $\beta = \mathrm{diag}\ _\sigma(\beta_\sigma)$, we have that $\gamma = \mathrm{diag}\ _\sigma(\gamma_\sigma) \in \mathrm{M}_{2n}(\mathbb{R})$ and

$$H_\mu(\beta v_0, \gamma v_0) = \sum_\sigma \mathrm{tr}\ (\mu^\sigma \beta_\sigma \iota_\sigma \bar{\gamma}_\sigma) + \sqrt{-1} \sum_\sigma \mathrm{tr}\ (\mu^\sigma \beta_\sigma \bar{\gamma}_\sigma).$$

Thus, if we let $H_\sigma \in \mathrm{GL}_2(\mathbb{C})$ denote the restriction of H_μ to $V_\sigma = \mathrm{M}_2(\mathbb{R}) \cdot \begin{pmatrix} \tau_\sigma & 1 \end{pmatrix}^t$, the matrix of H_μ respect to the chosen basis has diagonal form $H_\mu = \mathrm{diag}\ (H_\sigma)$.

In order to prove Theorem 5.1, it suffices to show that the hermitian form H_σ has $i_\sigma(\mu)$ negative eigenvalues. Take $\beta \in \mathrm{M}_2(\mathbb{R})$ and let $v = \beta \cdot \begin{pmatrix} \tau_\sigma & 1 \end{pmatrix}^t \in V_\sigma$. Then, $H_\sigma(v, v) = \mathrm{tr}(\mu^\sigma \beta_\sigma \iota_\sigma \bar{\beta}_\sigma) = \mathrm{tr}(\omega_\sigma \beta'_\sigma \iota'_\sigma \bar{\beta}'_\sigma)$, where $\beta'_\sigma = \nu_\sigma \beta_\sigma \nu_\sigma^{-1}$ and $\iota'_\sigma = \nu_\sigma \iota_\sigma \nu_\sigma^{-1}$. Denote $w_\sigma = \begin{pmatrix} w_1 & w_2 \end{pmatrix}^t = \nu_\sigma \beta_\sigma \cdot \begin{pmatrix} \tau_\sigma & 1 \end{pmatrix}^t \in \mathbb{C}^2$ and $\|w_\sigma\|^2 = w_1 \bar{w}_1 + w_2 \bar{w}_2$.

Some computation yields that

$$H_\sigma(v, v) = \sum_\sigma \frac{C_\sigma \sqrt{|\sigma(\delta)|}}{\det(\nu_\sigma) \mathrm{Im}(\tau_\sigma)},$$

where $C_\sigma = \|w\|^2$ if $\sigma(\delta) > 0$ and $C_\sigma = 2\mathrm{Re}(w_1 \bar{w}_2)$ if $\sigma(\delta) < 0$. From this, the result follows. □

Remark 5.2. From the above formula, the well-known relation $i(\mathcal{L}) + i(\mathcal{L}^{-1}) = \dim A$ ([21], Chapter III, Section 16, p. 150) is reobtained.

6. Principal polarizations and self-duality

If an abelian variety A admits a principal line bundle, and hence is self-dual, it is natural to ask whether it admits a principal polarization. It is the purpose of this section to study this question under the assumption that A has quaternionic multiplication by an hereditary order $\mathcal{O}$.

From Corollary 3.3, a sufficient condition for A to be self-dual when $(\vartheta_{F/\mathbb{Q}}, \mathcal{D}_\mathcal{O}) = 1$ is that $\mathcal{D}_\mathcal{O}$ and $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}}$ are principal ideals. By Theorem 5.1, a necessary condition for A to be principally polarizable is that $\mathcal{D}_\mathcal{O}$ be generated by a totally

positive element D in F. However, in general this is not sufficient for the existence of a principal polarization on A.

Let F_+^* denote the subgroup of totally positive elements of F^*, $R_{F_+}^* = R_F^* \cap F_+^*$ and $\mathcal{O}_+^* = \{\alpha \in \mathcal{O}^* : \mathrm{n}(\alpha) \in R_{F_+}^*\}$. Let $\mathrm{Pic}_+(F)$ be the narrow class group of F and let $h_+(F) = |\mathrm{Pic}_+(F)|$. We let $\Sigma = \Sigma(R_F^*) \subseteq \{\pm 1\}^n$ be the $\mathbb{F}_2$-subspace of signatures of units in R_F^*. As $\mathbb{F}_2$-vector spaces, $\Sigma \simeq R_F^*/R_{F+}^*$ and, by Dirichlet's Unit Theorem, $|\Sigma| = \dfrac{2^n h(F)}{h_+(F)}$. Since $\mathrm{n}(\mathcal{O}^*) = R_F^*$, the group Σ fits in the exact sequence

$$\begin{aligned} 1 \to \mathcal{O}_+^* \to \mathcal{O}^* \to \Sigma \to 1 \\ \alpha \mapsto (\mathrm{sgn}(\det \alpha^\sigma)). \end{aligned}$$

Definition 6.1. We denote by $\Omega \subseteq \{\pm 1\}^n$ the set of signatures

$$\Omega = \{(\mathrm{sgn}(\det \nu_\sigma(\mu)) : \mu \in \mathcal{P}(\mathcal{O})\}.$$

The set Ω can be identified with the set of connected components of $\mathbb{R}^n \backslash \cup_{i=1}^n \{x_i = 0\}$. With the notations as above, we obtain the following corollary of Theorems 3.2 and 5.1.

Proposition 6.2. *Let $\mathcal{I}$ be an ideal of B and assume that its left order $\mathcal{O}$ is hereditary. Then, there exist principally polarizable abelian varieties A with quaternionic multiplication by $\mathcal{O}$ and $H_1(A, \mathbb{Z}) \cong \mathcal{I}$ if and only if $\mathcal{D}_\mathcal{O}$ and $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}}$ are principal ideals and $\mathcal{D}_\mathcal{O} = (D)$ is generated by an element $D \in F_+^*$.*

If this is the case, an abelian variety $A = V/\mathcal{I} \cdot (\tau_1, 1, ..., \tau_n, 1)^t$, admits a principal polarization if and only if $(\mathrm{sgn}(\mathrm{Im}\tau_i)) \in \Omega$.

We note in passing that, as a consequence of the above corollary and an application of Čebotarev's Density Theorem ([23], Chapter VII, Section 13), self-dual but non principally polarizable abelian varieties A can be constructed. Examples of these abelian varieties are nontrivial, since in the generic case in which the ring of endomorphisms is $\mathrm{End}(A) = \mathbb{Z}$, the abelian variety A is principally polarizable if and only if it is self-dual.

Signature questions on number fields are delicate. In order to have a better understanding of Proposition 6.2, we describe Ω as the union (as sets) of linear varieties in the affine space $\mathbb{A}^n_{\mathbb{F}_2} = \{\pm 1\}^n$ as follows. Let $\{u_k\}$ be a set of representatives of units in R_F^*/R_F^{*2} and, for any order $S \supseteq R_F[\sqrt{-u_k D}]$ in $L = F(\sqrt{-u_k D})$, choose $\mu_S \in \mathcal{P}(S, \mathcal{O})$. We considered in Section 4 the Galois group $\Delta = \mathrm{Ker}(N : \mathrm{Pic}(S) \to \mathrm{Pic}(F))$. Naturally associated to it there is a sub-space of signatures $\Sigma(\Delta)$ in the quotient space $\mathbb{A}^n_{\mathbb{F}_2}/\Sigma(R_F^*)$: if $\mathfrak{b}$ is an ideal of S such that $N_{L/F}(\mathfrak{b}) = (b)$ for some $b \in F^*$, the signature of b does not depend on the choice of $\mathfrak{b}$ in its class in $\mathrm{Pic}(S)$ but depends on the choice of the generator b up to signatures in $\Sigma(R_F^*)$. By an abuse of notation, we still denote by $\Sigma(\Delta)$ the sub-space of $\mathbb{A}^n_{\mathbb{F}_2}$ generated by $\Sigma(R_F^*)$ and

the signatures of the norms of ideals in Δ. Then, from Proposition 4.4 we obtain that

$$\Omega = \sqcup_{k,S} \Sigma(\Delta) \cdot \mathrm{sgn}(\mu_S),$$

as a disjoint union. This allows us to compute the set Ω in many explicit examples and to show that, in many cases, coincides with the whole space of signatures $\{\pm 1\}^n$. The following corollary, which remains valid even if we remove the assumption $(\vartheta_{F/\mathbb{Q}}, \mathcal{D}_{\mathcal{O}}) = 1$, illustrates this fact.

Corollary 6.3. *Let F be a totally real number field of degree $[F : \mathbb{Q}] = n$. Let $\mathcal{O}$ be an hereditary order in a totally indefinite quaternion algebra B over F and let $\mathcal{I}$ be a left $\mathcal{O}$-ideal such that $\mathcal{D}_{\mathcal{O}} = (D)$ for $D \in F_+^*$ and $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}} = (t^{-1})$ for $t \in F^*$.*

If the narrow class number $h_+(F)$ of F equals the usual class number $h(F)$, i. e. if $\Sigma(R_F^*) = \{\pm 1\}^n$, then any abelian variety A with quaternionic multiplication by $\mathcal{O}$ and $H_1(A, \mathbb{Z}) \cong \mathcal{I}$ is principally polarizable.

In particular, if $h_+(F) = 1$, then the above conditions on $\mathcal{O}$ and $\mathcal{I}$ are accomplished.

Proof. Since $\Sigma(R_F^*) = \{\pm 1\}^n$, we have $\Omega = \{\pm 1\}^n$ and the result follows from Proposition 6.2. □

This is highly relevant in the study of certain *Shimura varieties*. As was already known to the specialists in the case of maximal orders, we obtain that *any* abelian surface with quaternionic multiplication by an *hereditary* order in an indefinite quaternion algebra $B/\mathbb{Q}$ admits a principal polarization.

7. The number of isomorphism classes of principal polarizations

Let $A = V/\Lambda$ with $\Lambda = \mathcal{I} \cdot v_0$ be an abelian variety with quaternionic multiplication by a maximal order $\mathcal{O}$. For any integer $0 \leq i \leq g$, let $\Pi_i(A)$ denote the set of isomorphism classes of principal line bundles $\mathcal{L} \in \mathrm{NS}(A)$ of index $i(\mathcal{L}) = i$. The set $\Pi(A)$ naturally splits as the disjoint union $\Pi(A) = \sqcup \Pi_i(A)$. Moreover, due to the relation $i(\mathcal{L}) + i(\mathcal{L}^{-1}) = g$, the map $\mathcal{L} \mapsto \mathcal{L}^{-1}$ induces a one-to-one correspondence between $\Pi_i(A)$ and $\Pi_{g-i}(A)$.

Formulas for $\pi_i(A)$, $0 \leq i \leq g$, analogous to that of Theorem 3.8 can be derived. Due to its significance, we will only concentrate on the number $\pi_0(A)$ of classes of principal polarizations. The Galois action on the sets $E(S, \mathcal{O})$ of Eichler classes of optimal embeddings and its behaviour respect to the index of the associated line bundles will play an important role.

Assume then that $\Pi_0(A) \neq \emptyset$. For simplicity, recall that we also assume that $(\vartheta_{F/\mathbb{Q}}, \mathcal{D}_{\mathcal{O}}) = 1$. By Proposition 6.2, we can choose $D \in F_+^*$ and $t \in F^*$ such that $\mathcal{D}_{\mathcal{O}} = (D)$ and $\mathrm{n}(\mathcal{I}) \cdot \vartheta_{F/\mathbb{Q}} = (t^{-1})$.

Let $u \in R_{F_+}^*$ be a totally positive unit. Let us agree to say that an order $S \supseteq R_F[\sqrt{-uD}]$ is *ample* respect to $\mathcal{O}$ if there exists an optimal embedding $i : S \hookrightarrow \mathcal{O}$

such that $\mu = i(\sqrt{-uD})$ is ample (cf. the discussion preceding Theorem 5.1). Define $\mathcal{S}_u$ to be the set of ample orders $S \supseteq R_F[\sqrt{-uD}]$ in $F(\sqrt{-uD})$. The existence of a principal polarization $\mathcal{L}$ on A implies that there is some $\mathcal{S}_u$ nonempty. With this notation, we obtain the following expression for $\pi_0(A)$ in terms of the narrow class number of F and the class numbers of certain CM-fields that embed in B.

Theorem 7.1. *The number of isomorphism classes of principal polarizations on A is*

$$\pi_0(A) = \frac{1}{2h_+(F)} \sum_{u \in R_{F_+}/R_F^{*2}} \sum_{S \in \mathcal{S}_u} 2^{e_S^+} h(S),$$

where $2^{e_S^+} = |R_{F+}^*/N(S^*)|$.

Proof. By the existing duality between $\Pi_0(A)$ and $\Pi_g(A)$, it is equivalent to show that $\pi_0(A) + \pi_g(A) = \sum_u \sum_{S \in \mathcal{S}_u} 2^{e_S^+} h(S)/h_+(F)$.

Let us introduce the set $\mathcal{P}_{0,g}(\mathcal{O}) = \{\mu \in \mathcal{O} : \mathrm{sgn}(\mu) = \pm(\mathrm{sign}(\mathrm{Im}\tau_i)), \mathrm{n}(\mu) \in R_{F_+}^* \cdot D\}$. By Theorems 2.2 and 5.1, the set $P_{0,g}(\mathcal{O}) = \mathcal{P}_{0,g}(\mathcal{O})^{\pm}/_{\sim_p}$ is in one-to-one correspondence with $\Pi_0(A) \cup \Pi_g(A)$ and we have a natural decomposition $P_{0,g}(\mathcal{O}) = \sqcup P_{0,g}(S, \mathcal{O})$ as S runs among ample orders in $\mathcal{S}_u$ and $u \in R_{F_+}^*/R_F^{*2}$.

Fix $u \in R_{F_+}^*$ and S in $\mathcal{S}_u$. In order to compute the cardinality of $P_{0,g}(S, \mathcal{O})$, we relate it to the set $E_{0,g}(S, \mathcal{O}) = \mathcal{P}_{0,g}(S, \mathcal{O})/_{\sim_e}$ of $\mathcal{O}_{\pm}^*$-Eichler conjugacy classes of optimal embeddings $i_\mu : S \hookrightarrow \mathcal{O}$. Here, we agree to say that two quaternions μ_1 and $\mu_2 \in \mathcal{P}_{0,g}(S, \mathcal{O})$ are Eichler conjugated by $\mathcal{O}_{\pm}^*$ if there is a unit $\alpha \in \mathcal{O}_{\pm}^*$ of either totally positive or totally negative reduced norm such that $\mu_1 = \alpha^{-1}\mu_2\alpha$. Note that, by Theorem 5.1, the action of $\mathcal{O}_{\pm}^*$-conjugation on the line bundle $\mathcal{L}$ associated to an element in $\mathcal{P}(S, \mathcal{O})$ either preserves the index $i(\mathcal{L})$ or switches it to $g - i(\mathcal{L})$. This makes sense of the quotient $E_{0,g}(S, \mathcal{O})$.

We have the following exact diagram:

$$\begin{array}{ccccccccc} 0 & \to & \Delta_+ & \to & \mathrm{Pic}(S) & \overset{N_{L/F}}{\to} & \mathrm{Pic}_+(F) & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & \Delta & \to & \mathrm{Pic}(S) & \overset{N_{L/F}}{\to} & \mathrm{Pic}(F) & \to & 0. \end{array}$$

Indeed, there is a natural map $\mathrm{Pic}(S) \overset{N_{L/F}}{\to} \mathrm{Pic}_+(F)$, since the norm of an element $a + b\sqrt{-uD} \in L$ for a, $b \in F$ is $a^2 + ub^2D \in F_+^*$. The surjectivity of the map $\mathrm{Pic}(S) \to \mathrm{Pic}_+(F)$ is argued as in Section 4 by replacing the Hilbert class field H_F of F by the big Hilbert class field H_F^+, whose Galois group over F is $\mathrm{Gal}(H_F^+/F) = \mathrm{Pic}_+(F)$. By Proposition 4.4, Δ acts freely and transitively on $E(S, \mathcal{O})$. Therefore, by Theorem 5.1, there is also a free action of Δ_+ on $E_{0,g}(S, \mathcal{O})$. Up to sign, the $\mathcal{O}_{\pm}^*$-Eichler conjugation class of an element $\mu \in \mathcal{P}(S, \mathcal{O})$ has a well-defined orientation $\pm\mathrm{sgn}(\mu)$. Note also that two inequivalent $\mathcal{O}_{\pm}^*$-Eichler classes that fall in the same

$\mathcal{O}^*$-conjugation class are never oriented in the same manner, even not up to sign. Taken together, this shows that Δ_+ also acts transitively on $E_{0,g}(S,\mathcal{O})$. This means that

$$|E_{0,g}(S,\mathcal{O})| = \frac{h(S)}{h_+(F)}.$$

There is again a natural surjective map $\rho : P_{0,g}(S,\mathcal{O}) \to E_{0,g}(S,\mathcal{O})$ and, arguing as in Section 3, Theorem 7.1 follows. □

Examples in low dimensions. When we particularize our results to dimension 2, we obtain an easy to apply expression for the number of principal polarizations on an abelian surface with maximal quaternionic multiplication, as it is stated in Theorem 1.1 in the introduction. As an example, the number of isomorphism classes of principal polarizations on an abelian surface A with quaternionic multiplication by a maximal order in a quaternion algebra of discriminant $D = 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19$ is $\pi_0(A) = 1040$. This also implies the existence of 1040 pairwise nonisomorphic smooth algebraic curves $C_1, ..., C_{1040}$ of genus 2 such that their Jacobian varieties are isomorphic as unpolarized abelian surfaces.

In addition, since $\pi(A) = \pi_0(A) + \pi_1(A) + \pi_2(A)$ and $\pi_0(A) = \pi_2(A)$, Theorems 3.8 and 7.1 yield the formula

$$\pi_1(A) = \begin{cases} \varepsilon_{4D}\cdot h(4D) + \varepsilon_D \cdot h(D) & \text{if } D \equiv 1 \mod 4, \\ \varepsilon_{4D}\cdot h(4D) & \text{otherwise,} \end{cases}$$

where ε_D and $\varepsilon_{4D} = 1$ or $\frac{1}{2}$ is computed from the formula for e_S in Theorem 3.8.

Let F be the real quadratic field $\mathbb{Q}(\sqrt{2})$ and let B be the quaternion algebra over F that ramifies exactly at the two prime ideals $(3\pm\sqrt{2})$ above 7. By applying Theorems 3.8, 5.1 and 7.1 and the valuable help of the programming package PARI ([24]), we conclude that, for any abelian four-fold A such that $\mathrm{End}(A)$ is a maximal order in the quaternion algebra $B/\mathbb{Q}(\sqrt{2})$ of discriminant 7, the number of isomorphism classes of principal line bundles of index 0, 1, 2, 3 and 4 are $\pi_0(A) = \pi_4(A) = 6$, $\pi_1(A) = \pi_3(A) = 4$ and $\pi_2(A) = 4$, respectively.

8. **Asymptotic behaviour of $\pi_0(A)$**

We can combine Theorem 7.1 with analytical tools to estimate the asymptotic behaviour of $\log(\pi_0(A))$. This will yield a stronger version of part 1 of Theorem 1.2 in the introduction. For any number field L, we let D_L and Reg_L stand for the absolute value of the discriminant and the regulator, respectively.

Theorem 8.1. *Let F be a totally real number field of degree n. Let A range over a sequence of principally polarizable abelian varieties with quaternionic multiplication*

by a maximal order in a totally indefinite quaternion algebra B over F of discriminant $D \in F_+^*$ with $|N_{F/\mathbb{Q}}(D)| \to \infty$. Then

$$\log \pi_0(A) \sim \log \sqrt{|N_{F/\mathbb{Q}}(D)| \cdot D_F}.$$

The proof of Theorem 8.1 adapts an argument of Horie-Horie ([12]) on estimates of relative class numbers of CM-fields. We first show that there indeed exist families of abelian varieties satisfying the properties quoted in the theorem.

By Čebotarev's Density Theorem, we can find infinitely many pairwise different totally positive principal prime ideals $\{\wp_i\}_{i\geq 1}$ in F. We can also choose them such that $(\wp_i, \vartheta_{F/\mathbb{Q}}) = 1$. We then obtain principal ideals $(D_j) = \wp_1 \cdot \wp_2 \cdot \ldots \cdot \wp_{2j-1} \cdot \wp_{2j}$ with $D_j \in F_+^*$ and $(D_j, \vartheta_{F/\mathbb{Q}}) = 1$. According to [31], p. 74, there exists a totally indefinite quaternion algebra B_j over F of discriminant D_j for any $j \geq 1$. Then, Proposition 6.2 asserts that there exists an abelian variety A_j of dimension $2n$ such that $\mathrm{End}(A_j)$ is a maximal order in B_j and $\Pi_0(A_j) \neq \emptyset$.

Proof of Theorem 8.1. Let A be a principally polarizable abelian variety with quaternionic multiplication by a maximal order in a totally indefinite division quaternion algebra B over F of discriminant $D \in F_+^*$.

For any totally positive unit $u_k \in R_{F_+}^*$, let $L_k = F(\sqrt{-u_k D})$. For any order $S \supseteq R_F[\sqrt{-u_k D_j}]$ in the CM-field L_k, it holds that $h(S) = c_S h(L_k)$ for some positive constant $c_S \in \mathbb{Z}$ which is uniformly bounded by 2^n.

The class number $h(F)$ turns out to divide $h(L_k)$ and the *relative class number* of L_k is defined to be $h^-(L_k) = h(L_k)/h(F)$ (cf. [19]). Since $h_+(F) = 2^m h(F)$ for $m = n - \dim_{\mathbb{F}_2}(\Sigma(R_F^*))$, Theorem 7.1 can be rephrased as $\pi_0(A) = \sum 2^{(e_S^+ - 1 - m)} c_S h^-(L_k)$.

In order to apply the Brauer-Siegel Theorem, the key point is to relate the several absolute discriminants D_{L_k} and regulators Reg_{L_k} as u_k vary among totally positive units in F. Firstly, we have the relations $D_{L_k} = |N_{F/\mathbb{Q}}(D_{L_k/F}) \cdot D_F^2| = 2^{p_k} |N_{F/\mathbb{Q}}(D)| D_F^2$ for some $0 \leq p_k \leq 2n$. Secondly, by [32], p. 41, it holds that $\mathrm{Reg}_{L_k} = 2^c \mathrm{Reg}_F$ with $c = n-1$ or $n-2$.

Let ε be a sufficiently small positive number. By the Brauer-Siegel Theorem, it holds that $D_{L_k}^{(1-\varepsilon)/2} \leq h(L_k)\mathrm{Reg}_{L_k} \leq D_{L_k}^{(1+\varepsilon)/2}$ for $D_{L_k} \gg 1$. Thus

$$\frac{D_F^{(1-\varepsilon)/2}}{h(F)\mathrm{Reg}_F}(D_{L_k}/D_F)^{(1-\varepsilon)/2} \leq h^-(L_k) \leq \frac{D_F^{(1+\varepsilon)/2}}{h(F)\mathrm{Reg}_F}(D_{L_k}/D_F)^{(1+\varepsilon)/2}.$$

Fixing an arbitrary CM-field L in the expression for $\pi_0(A)$, this boils down to

$$C_- \cdot \frac{D_F^{(1-\varepsilon)/2}}{h(F)\mathrm{Reg}_F}(D_L/D_F)^{(1-\varepsilon)/2} \leq \pi_0(A) \leq C_+ \cdot \frac{D_F^{(1+\varepsilon)/2}}{h(F)\mathrm{Reg}_F}(D_L/D_F)^{(1+\varepsilon)/2}$$

for some positive constants C_- and C_+. Taking logarithms, these inequalities yield Theorem 8.1. □

Remark 8.2. The argument above is not effective since it relies on the classical Brauer-Siegel Theorem on class numbers. However, recent work of Louboutin ([19], [20]) on lower and upper bounds for relative class numbers of CM-fields, based upon estimates of residues at $s = 1$ of Dedekind zeta functions, could be used to obtain explicit lower and upper bounds for $\pi_0(A)$.

Finally, we conclude this paper with the proof of the second main result quoted in the Introduction.

Proof of Theorem 1.2. Part 1 is an immediate consequence of Theorem 8.1. Let us explain how part 2 follows. Assume that A is a simple complex abelian variety of odd and square-free dimension g. Then, by Albert's classification of simple division algebras ([21], Chapter IV, Section 19 and 21), $\mathrm{End}(A) \simeq S$ is an order in either a totally real number field F or a CM-field L over a totally real number field F. In any case, $[F : \mathbb{Q}] \leq g$. In the former case, by Theorem 3.1 of Lange in [17], $\pi_0(A) = |S_+^*/S^{*2}| \leq 2^{g-1}$. In the latter, let $S_0 \subset F$ be the subring of S fixed by complex conjugation. If $\mathcal{L}$ is a principal polarization on A, the Rosati involution precisely induces complex conjugation on $\mathrm{End}(A) \simeq S$ and we have that $\pi_0(A) = |S_{0+}^*/\mathrm{N}_{L/F}(S^*)| \leq |S_{0+}^*/S_0^{*2}| \leq 2^{g-1}$, by applying Theorem 1.5 of [17].

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