On the Alexandroff-Bakelman-Pucci Estimate and the Reversed Hölder Inequality for Solutions of Elliptic and Parabolic Equations

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Abstract

The constant appearing in the classical Alexandroff-Bakelman-Pucci estimate for subsolutions of second-order uniformly elliptic equations in nondivergence form was known to depend on the diameter of the domain. Using the Krylov-Safonov boundary weak Harnack inequality due to Trudinger, we show that the dependence on the diameter may be replaced by dependence on a more precise geometric quantity of the domain. As a consequence, we get dependence on the measure instead of the diameter. We also give new bounds for subsolutions in some unbounded domains, such as domains contained in cones.

We apply the Fabes and Stroock reversed Hölder inequality for the Green's function to improve our estimates. We also give a new proof of the reversed Hölder inequality for the Green's function based on the Krylov-Safonov Harnack inequality.

Finally, we find new bounds for subsolutions of uniformly parabolic equations in cylindrical and noncylindrical domains. The constant in the (Alexandroff-Bakelman-Pucci-) Krylov-Tso estimate was known to depend on the diameter of the base of the cylinder. We get dependence either on the measure of the base or on the height of the cylinder. We also give bounds for subsolutions in noncylindrical domains. © 1995 John Wiley & Sons, Inc.

Summary

Let L be a second-order uniformly elliptic operator in a domain $\Omega \subset \mathbb{R}^n$, of the form

$$Lw = a_{ij}(x)\partial_{ij}w + b_i(x)\partial_i w + c(x)w\,.$$

Let w be smooth and bounded above in $\bar{\Omega}$ and $f \in L^n(\Omega)$ satisfy

$$Lw \geqq f \ \text{ in } \Omega \quad \text{and} \quad w \leqq 0 \ \text{ on } \partial\Omega\,.$$

In the case that Ω is bounded and $c \leqq 0$ in Ω, the classical Alexandroff-Bakelman-Pucci estimate, which we call ABP estimate, asserts that

$$\sup_{\Omega} w \leqq C \operatorname{diam}(\Omega)\, ||f||_{L^n(\Omega)}\,.$$

Here $\operatorname{diam}(\Omega)$ denotes the diameter of Ω and C (as all constants C in our results) depends essentially only on the ellipticity and L^∞-norms of the coefficients of L.

We use the Krylov-Safonov boundary weak Harnack inequality to improve the ABP estimate as follows (see Theorem 1.4 for a more general result). Assume that $R > 0$ and $0 < \sigma < 1$ are numbers such that for any $x \in \Omega$ there is an

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open ball B_{R_x} (not necessarily centered at x) of radius $R_x \leqq R$ satisfying $x \in B_{R_x}$ and $|B_{R_x} \setminus \Omega| \geqq \sigma |B_{R_x}|$, where $|\cdot|$ denotes the Lebesgue measure. Under these assumptions we prove that

$$\sup_{\Omega} w \leqq C R \, ||f||_{L^n(\Omega)} .$$

Here Ω may be unbounded and C depends also on σ. We still suppose $c \leqq 0$. As a corollary we get that in the ABP estimate diam(Ω) may be replaced by $|\Omega|^{1/n}$.

We also consider domains Ω which satisfy $|A(r,2r) \setminus \Omega| \geqq \bar{\sigma} |A(r,2r)|$ for any $r > R_1$, where $0 < \sigma < 1$ and $R_1 \geqq 0$ are constants and $A(r,2r) = \{x \in \mathbb{R}^n : r < |x| < 2r\}$. Under these assumptions, we show that

$$\sup_{\Omega} w \leqq C \, || \, (R_1 + |x|) f(x) \, ||_{L^n(\Omega)} .$$

In particular, this estimate holds, with $R_1 = 0$, in any domain contained in a proper closed cone of $\mathbb{R}^n$.

We give a new proof of the Fabes and Stroock reversed Hölder inequality for the Green's function; our proof is based on the Krylov-Safonov Harnack inequality. We use a consequence of the reversed Hölder inequality for the Green's function to replace $||f||_{L^n(\Omega)}$ by $||f||_{L^{p_0}(\Omega)}$ (for some $p_0 < n$ depending on the ellipticity of L) in our improved ABP estimate and in the Krylov-Safonov Harnack inequality.

Finally, let L be now a second-order uniformly parabolic operator in a domain $D \subset \mathbb{R}^{n+1}$ (D need not be bounded nor cylindrical), of the form

$$Lw = -\partial_t w + a_{ij}(x,t)\partial_{ij} w + b_i(x,t)\partial_i w + c(x,t) w .$$

Let w be bounded above in D satisfy $Lw \geqq f$ in D and $w \leqq 0$ on $\partial_p D$. Assume that $c \leqq 0$ in D. In the case that $D \subset B_d \times (0,\infty)$ for some ball B_d of radius d in $\mathbb{R}^n$, the ABP-Krylov-Tso estimate asserts that

$$\sup_{D} w \leqq C \, d^{\,n/(n+1)} \, ||f||_{L^{n+1}(D)} .$$

We improve this estimate in a similar way as in the elliptic case; in particular we replace $d^{\,n/(n+1)}$ by

(i) $|D|^{\,n/((n+1)(n+2))}$ if D has finite $(n+1)$-dimensional Lebesgue measure $|D|$,

(ii) $|\Omega|^{1/(n+1)}$ if $D \subset \Omega \times (0,\infty)$ and Ω has finite n-dimensional Lebesgue measure $|\Omega|$, and

(iii) $T^{\,n/(2(n+1))}$ if $D \subset \mathbb{R}^n \times (0,T)$.

1. Introduction and Results

Part (a). The Elliptic Case

Let L be an elliptic operator in a domain $\Omega \subset \mathbb{R}^n$, of the form

$$Lw = Mw + c(x)w = a_{ij}(x)\partial_{ij} w + b_i(x)\partial_i w + c(x) w , \tag{1.1}$$

which is uniformly elliptic, i.e., $a_{ij} = a_{ji}$ are measurable, not necessarily continuous, functions in Ω and

$$c_0|\xi|^2 \leqq a_{ij}(x)\xi_i\xi_j \leqq C_0|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^n \quad \forall\, x \in \Omega \tag{1.2}$$

for some positive constants c_0 and C_0.

Unless otherwise stated, Ω may be either bounded or unbounded. We assume that $b_i \in L^\infty(\Omega)$ and we fix a positive constant b such that

$$\left(\sum b_i(x)^2\right)^{1/2} \leqq b \qquad \forall\, x \in \Omega\,. \tag{1.3}$$

We also assume that c is a measurable, not necessarily bounded, function in Ω.

L will always act on functions in $W^{2,p}_{\mathrm{loc}}(\Omega)$, the Sobolev space of functions which belong, together with their second derivatives, to $L^p_{\mathrm{loc}}(\Omega)$. We will always have $p > n/2$, and hence functions in $W^{2,p}_{\mathrm{loc}}(\Omega)$ will be continuous in Ω.

Let us denote by $\mathrm{diam}(\Omega)$ and $|\Omega|$, the diameter and the measure, respectively, of Ω (in case they are finite). Given a function w, we denote by w^+ and w^- the positive and negative parts of w, respectively, so that $w = w^+ - w^-$. If $B_R = B_R(x_0)$ is a ball (by ball we always mean open ball) of radius R in $\mathbb{R}^n$, we denote $B_{\tau R}(x_0)$ by $B_{\tau R}$.

Under these assumptions we have the following basic estimate for subsolutions of $Lw = f$ (see Section 9.1 in [6]), which we call ABP estimate.

THEOREM 1.1. (ALEXANDROFF, BAKELMAN, PUCCI) *Let Ω be bounded and d be a positive constant such that* $\mathrm{diam}(\Omega) \leqq d$. *Assume that $c \leqq 0$ in Ω,*

$$w \in W^{2,n}_{\mathrm{loc}}(\Omega)\,, \quad f \in L^n(\Omega) \quad \textit{and} \quad Lw \geqq f \textit{ in } \Omega\,.$$

Then

$$\sup_\Omega w \leqq \limsup_{x\to\partial\Omega} w^+(x) + C\,\mathrm{diam}(\Omega)\,||f||_{L^n(\Omega)}\,, \tag{1.4}$$

where C is a constant depending only on n, c_0, and bd.

Note that the ABP estimate applies only to operators L with $c \leqq 0$. The paper by Berestycki, Nirenberg, and Varadhan, [2], however, contains an extension of the ABP estimate to operators L for which the principal eigenvalue of $-L$ is positive; see Theorem 1.3 in [2].

The constant C in the ABP estimate depends essentially only on the ellipticity and L^∞-norms of the coefficients of L. All constants in our results will also depend only on these quantities. This fact has the important consequence that the ABP estimate and our new estimates also apply to solutions of second-order fully nonlinear uniformly elliptic equations. In this direction, the ABP estimate is a very useful tool in the study of fully nonlinear equations; see [3].

In this paper we improve the ABP estimate by replacing the factor $\mathrm{diam}(\Omega)$ by a more precise geometric quantity of Ω. In particular we show that $\mathrm{diam}(\Omega)$ may

be replaced by $|\Omega|^{1/n}$. The new geometric constant, that already appears — in an essentially equivalent form — in Theorem 2.5 of [2], is the following.

DEFINITION 1.2. Let $0 < \sigma < 1$ and $0 < \tau < 1$ be fixed numbers and Ω be a domain. We say that a positive number R satisfies condition (G) relative to σ, τ, Ω, if for any $x \in \Omega$ there exists a ball B_{R_x} of radius R_x in $\mathbb{R}^n$ such that $R_x \leqq R$ and

$$x \in B_{R_x} \quad \text{and} \quad |B_{R_x} \setminus \Omega_x| \geqq \sigma |B_{R_x}| \,, \tag{1.5}$$

where Ω_x is the component of $B_{R_x/\tau} \cap \Omega$ containing x. Note that we do not assume B_{R_x} to be centered at x.

For a bounded domain Ω, condition (G) is always satisfied for R large enough (see Remark 1.3). In Remark 2.1 we give some examples of unbounded domains for which there is some R satisfying condition (G). We also make some observations about condition (G) in the beginning of Section 2.

Remark 1.3. Assume that Ω has finite measure and let R be the positive constant such that $|B_R| = 2|\Omega|$, where B_R is any ball in $\mathbb{R}^n$ of radius R. It follows that $R = C(n)|\Omega|^{1/n}$, for some constant $C(n)$ depending only on n, and taking $B_{R_x} = B_R(x)$ in (1.5), we have that $|B_R(x) \setminus \Omega_x| \geqq |B_R(x) \setminus \Omega| \geqq |B_R(x)| - |\Omega| = |B_R(x)|/2$.

We conclude that R satisfies condition (G) relative to $1/2, 1/2, \Omega$.

We now state our main result, which we call improved ABP estimate. We replace $\operatorname{diam}(\Omega)$ in the ABP estimate by R, where R is as in Definition 1.2. We recall that Ω may be either bounded or unbounded.

THEOREM 1.4. *Assume that R satisfies condition* (G) *relative to σ, τ, Ω (see Definition* 1.2) *and $R \leqq R_0$, for some constants σ, τ and R_0. Assume also that $c \leqq 0$ in Ω,*

$$w \in W^{2,n}_{\mathrm{loc}}(\Omega)\,, \quad f \in L^n(\Omega) \quad \textit{and} \quad Lw \geqq f \ \textit{ in } \Omega\,. \tag{1.6}$$

Suppose that w is bounded above in Ω. Then

$$\sup_{\Omega} w \leqq \limsup_{x \to \partial\Omega} w^+(x) + C\,R\,||f||_{L^n(\Omega)}\,, \tag{1.7}$$

where C is a constant depending only on $n, c_0, C_0, bR_0, \sigma$, and τ.

As an immediate consequence of Theorem 1.4 and Remark 1.3 we conclude the following.

COROLLARY 1.5. *Assume $|\Omega| \leqq m$ for a positive constant m, $c \leqq 0$ in Ω and* (1.6). *Suppose that w is bounded above in Ω. Then*

$$\sup_{\Omega} w \leqq \limsup_{x \to \partial\Omega} w^+(x) + C\,|\Omega|^{1/n}\,||f||_{L^n(\Omega)}\,, \tag{1.8}$$

where C is a constant depending only on n, c_0, C_0, and $bm^{1/n}$.

In [2], the authors stated estimate (1.8) as an open question, suggested by the following result (Theorem 10.1 in [2]). *If Ω is bounded, $c \leqq 0$ in Ω and $Lw \geqq f$ in Ω then*

$$\sup_{\Omega} w \leqq \limsup_{x \to \partial\Omega} w^+(x) + C\ |\Omega|^{2/n}\ ||f||_{L^\infty(\Omega)}\ .$$

The improved ABP estimate will follow easily from a deep result called the Krylov-Safonov boundary weak Harnack inequality (see Theorem 2.2), which is due to Trudinger and is essentially equivalent to the Krylov-Safonov Harnack inequality (see Theorems 9.20 and 9.22 in [6]). The proof of the boundary weak Harnack inequality makes use of the ABP estimate; in this sense our result is not independent of the ABP estimate.

Theorem 1.4 is also true for viscosity subsolutions. The proof of this fact does not differ substantially from the one we give in this paper for strong subsolutions.

Most of the results in this paper hold in the case that the coefficients b_i and/or c are not bounded but belong to suitable $L^p(\Omega)$ spaces (see Remark 3.3).

We point out that analogous estimates to the ones in this paper — like our improved ABP estimate — hold for solutions of elliptic equations in divergence form. In this case $||f(x)||_{L^n(\Omega)}$ is replaced by $||f(x)||_{L^p(\Omega)}$ for any $p > n/2$. This will be studied in more detail in a future paper.

We now describe an application of Theorem 1.4 to the maximum principle for operators L with $c(x)$ of arbitrary sign.

Definition 1.6. We say that the maximum principle holds for the operator L in Ω if $w \in W^{2,n}_{\mathrm{loc}}(\Omega)$,

$$\sup_{\Omega} w < \infty\ , \quad Lw \geqq 0 \ \text{ in } \Omega \quad \text{and} \quad \limsup_{x \to \partial\Omega} w(x) \leqq 0$$

imply $w \leqq 0$ in Ω.

Note that, in the case that Ω is bounded, the assumption that w is bounded above follows from the other assumptions on w. By the ABP estimate, $c \leqq 0$ is a sufficient condition — by no means necessary — for the maximum principle to hold for L in any bounded domain Ω.

In [2] new sufficient conditions for the maximum principle to hold (for an operator L with $c(x)$ of arbitrary sign) are established in the case of general bounded domains. These sufficient conditions have important applications to the study of some linear and nonlinear problems; for instance, symmetry and monotonicity properties of solutions of elliptic equations; see [9]. We will use our improved ABP estimate to give an alternative proof of the following strong sufficient condition given in [2]. It is essentially Theorem 2.5 in [2]. It states that the maximum principle holds for L in Ω if a sufficiently small R satisfies condition (G) relative to $1/2, 1/2, \Omega$. R needs to be small enough depending only on the ellipticity and

L^∞-norms of the coefficients of L (including $c(x)$). We point out again that, in addition, we allow Ω to be unbounded.

COROLLARY 1.7. (BERESTYCKI, NIRENBERG, VARADHAN) *Let $0 < \sigma < 1$ be a fixed constant and assume that $c \leqq b'$ in Ω, for a positive constant b'. Then there exists a positive constant R^*, which depends only on n, c_0, C_0, b, b', and σ, such that if R^* satisfies condition* (G) *relative to $\sigma, 1/2, \Omega$ then the maximum principle holds for L in Ω.*

In fact, the proof of this corollary will give that the maximum principle holds for L in Ω if $(R^*)^2 b'$ is small enough depending only on n, c_0, C_0, b, and σ. In Remark 2.4 we give some examples in which Corollary 1.7 applies.

The method used in the proof of the improved ABP estimate (Theorem 1.4) will also give the following bound. It applies in some unbounded domains for which there is no R satisfying condition (G). For instance, the bound holds in any unbounded domain contained in a proper closed cone of $\mathbb{R}^n$ (the cone need not be convex, nor spherical).

Let us denote, for any $0 < r < r'$,

$$A(r, r') = \{ x \in \mathbb{R}^n : r < |x| < r' \} .$$

THEOREM 1.8. *Let Ω be a domain of $\mathbb{R}^n$ such that*

$$|A(r, 2r) \setminus \Omega| \geqq \sigma |A(r, 2r)| \qquad \forall\, r > R_1 , \tag{1.9}$$

for some fixed constants $0 < \sigma < 1$ and $R_1 \geqq 0$. Instead of (1.3), *assume*

$$(R_1 + |x|) \left(\sum b_i(x)^2 \right)^{1/2} \leqq B \qquad \forall\, x \in \Omega ,$$

for a positive constant B. Assume also that $c \leqq 0$ in Ω,

$$w \in W^{2,n}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad Lw \geqq f \ \text{ in } \Omega .$$

Suppose that w is bounded above in Ω. Then

$$\sup_\Omega w \leqq \limsup_{x \to \partial\Omega} w^+(x) + C \, || \, (R_1 + |x|) f(x) \, ||_{L^n(\Omega)} , \tag{1.10}$$

where C is a constant depending only on n, c_0, C_0, B, and σ.

In Section 4 we give a new proof of the Fabes and Stroock reversed Hölder inequality for the Green's function (see Theorems 4.1 and 4.2). Our proof is based on the Krylov-Safonov Harnack inequality (see Corollary 9.25 in [6]). We point out that in [4] the reversed Hölder inequality for the Green's function was used to give an alternative proof of the Krylov-Safonov Harnack inequality. Therefore it turns out that these two inequalities are essentially equivalent (modulo the

ABP estimate) as strong results in the theory of second-order nondivergence form elliptic operators.

In [4] Fabes and Stroock use their reversed Hölder inequality and a result by Gehring to improve the ABP estimate. We state this result in Corollary 5.1 and we use it to improve estimate (1.7) in the following way. We recall that Ω may be unbounded.

THEOREM 1.9. *Let $n \geqq 2$ and assume that R satisfies condition* (G) *relative to σ, τ, Ω, and $R \leqq R_0$, for some constants σ, τ, and R_0. Suppose that $c \leqq 0$ in Ω. Then there exists $p_0 \in (n/2, n)$ depending only on n, c_0, C_0, and bR_0/τ such that*

$$w \in W^{2,p_0}_{\mathrm{loc}}(\Omega)\,, \quad f \in L^{p_0}(\Omega)\,, \quad Lw \geqq f \ \text{ in } \Omega\,,$$

and w bounded above in Ω imply

$$\sup_{\Omega} w \leqq \limsup_{x \to \partial\Omega} w^+(x) + CR^{2-n/p_0}\, ||f||_{L^{p_0}(\Omega)}\,, \tag{1.11}$$

where C is a constant depending only on $n, c_0, C_0, bR_0, \sigma$, and τ. In particular, we have that

$$w \in W^{2,p_0}_{\mathrm{loc}}(\Omega)\,, \quad \sup_{\Omega} w < \infty\,, \quad Lw \geqq 0 \ \text{ in } \Omega \ \text{ and } \ \limsup_{x \to \partial\Omega} w(x) \leqq 0$$

imply $w \leqq 0$ in Ω.

Remark 1.10. The examples following Theorem 8.9 in [6] show that, with p_0 as in the previous theorem, $p_0 \geqq n[1 + (n-1)c_0/C_0]^{-1}$. Therefore $p_0 \to n$ as $c_0/C_0 \to 0$.

The last part of the previous theorem states that if $c \leqq 0$ in Ω then the maximum principle holds for L in Ω for functions in $W^{2,p_0}_{\mathrm{loc}}(\Omega)$. In particular the maximum principle holds for functions that together with their second derivatives belong to $L^{n,\mathrm{weak}}_{\mathrm{loc}}(\Omega)$ — the weak $L^n_{\mathrm{loc}}(\Omega)$ space. We point out that up to now the maximum principle was only known to hold for functions in $W^{2,n}_{\mathrm{loc}}(\Omega)$.

We also improve the Krylov-Safonov Harnack inequality (see Theorems 9.20 and 9.22 in [6]) as follows.

THEOREM 1.11. *Assume that $n \geqq 2$, $c \in L^\infty(\Omega)$, and $|c| \leqq \tilde{b}$ in Ω, for a positive constant $\tilde{b}$ (here we make no assumption on the sign of c). Let B_R be a ball of radius R in $\mathbb{R}^n$ such that $B_{2R} \subset \Omega$ and $R \leqq R_0$, for some positive constant R_0. Then there exists $p_0 \in (n/2, n)$ depending only on n, c_0, and C_0 such that*

$$u \in W^{2,p_0}(\Omega)\,, \quad Lu = f \ \text{ in } \Omega \ \text{ and } \ u \geqq 0 \ \text{ in } \Omega$$

imply

$$\sup_{B_R} u \leqq C\left(\inf_{B_R} u + R^{2-n/p_0}\, ||f||_{L^{p_0}(B_{2R})}\right), \tag{1.12}$$

where C is a constant depending only on n, c_0, C_0, bR_0, and $\tilde{b}R_0^2$.

Part (b). The Parabolic Case

We denote here some objects with the same notation as in the elliptic case. There is no risk of confusion since Section 6 is the only other part of this article where we deal with the parabolic case.

Let L be a parabolic operator in a domain $D \subset \mathbb{R}^{n+1} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}\}$, of the form

$$Lw = Mw + c(x,t)w = -\partial_t w + a_{ij}(x,t)\partial_{ij}w + b_i(x,t)\partial_i w + c(x,t)w\,, \tag{1.13}$$

such that $a_{ij} = a_{ji}$ are measurable, not necessarily continuous, functions in D and

$$c_0|\xi|^2 \leqq a_{ij}(x,t)\xi_i\xi_j \leqq C_0|\xi|^2 \qquad \forall\ \xi \in \mathbb{R}^n \quad \forall\ (x,t) \in D \tag{1.14}$$

for some positive constants c_0 and C_0.

Unless otherwise stated, D may be either bounded or unbounded. We do not assume D to be cylindrical. We assume that $b_i \in L^\infty(D)$ and we fix a positive constant b such that

$$\left(\sum b_i(x,t)^2\right)^{1/2} \leqq b \qquad \forall\ (x,t) \in D\,. \tag{1.15}$$

We also assume that c is a measurable, not necessarily bounded, function in D.

L will always act on functions in the Sobolev space $W^{2,1}_{n+1,\mathrm{loc}}(D)$. We define this space in Section 6.

Let us consider, for $(x_0, t_0) \in \mathbb{R}^{n+1}$,

$$Q_R(x_0,t_0) = B_R(x_0) \times (t_0 - R^2, t_0)\,.$$

$Q_R(x_0,t_0)$ will be called the parabolic cylinder of radius R with top center at (x_0,t_0). If $Q_R = Q_R(x_0,t_0)$, we denote $Q_{\tau R}(x_0,t_0)$ by $Q_{\tau R}$; note that Q_R and $Q_{\tau R}$ have (x_0,t_0) as the same top center.

Let us denote by $|D|$ the $(n+1)$-dimensional Lebesgue measure of D (in case it is finite) and by $\partial_{\mathrm{p}}D$ the parabolic boundary of D. The concept of parabolic boundary of a noncylindrical domain will be reviewed in Section 6.

Under these assumptions we have the following estimate for subsolutions of $Lw = f$ (see Theorem 9 in Section 3.3 of [7]), which we call ABP-Krylov-Tso estimate.

THEOREM 1.12. (KRYLOV) *Assume that $D \subset B_d \times (0,\infty)$, where d is a positive constant and B_d is a ball in $\mathbb{R}^n$ of radius d, and*

$$c \leqq 0 \text{ in } D\,,\quad w \in W^{2,1}_{n+1,\mathrm{loc}}(D) \cap C(\overline{D})\,,\quad f \in L^{n+1}(D)\,,\quad Lw \geqq f \text{ in } D\,. \tag{1.16}$$

Then

$$\sup_D w \leqq \sup_{\partial_{\mathrm{p}}D} w^+ + C\, d^{\,n/(n+1)}\, ||f||_{L^{n+1}(D)}\,, \tag{1.17}$$

where C is a constant depending only on n, c_0, and bd.

We point out that in the case of a cylindrical domain $D = \Omega \times (0,T)$, the ABP-Krylov-Tso estimate is also proved in [12] (it is Theorem 2.1 of [12]).

As in the elliptic case we introduce the following geometric quantity of the domain D.

DEFINITION 1.13. Let $0 < \tau \leqq \sigma < 1$ be fixed numbers and D be a domain of $\mathbb{R}^{n+1}$. We say that a positive number R satisfies condition (PG) relative to σ, τ, D, if for any $(x,t) \in D$ there exists a parabolic cylinder $Q_{R_{(x,t)}}$ of radius $R_{(x,t)}$ such that $R_{(x,t)} \leqq R$ and

$$(x,t) \in Q_{\tau R_{(x,t)}} \quad \text{and} \quad |Q_{R_{(x,t)}} \setminus D_{(x,t)}| \geqq \sigma |Q_{R_{(x,t)}}| \,, \tag{1.18}$$

where $D_{(x,t)}$ is the component of $Q_{R_{(x,t)}} \cap D$ containing (x,t).

Remark 1.14.

(i) Assume that $|D| < \infty$ and let R be the positive constant such that $|Q_R| = 2|D|$, where Q_R is any parabolic cylinder of radius R. Then $R = C(n)|D|^{1/(n+2)}$, for a positive constant $C(n)$ depending only on n, and R satisfies condition (PG) relative to $1/2, 1/2, D$.

(ii) Assume that

$$D \subset \Omega \times (0,\infty) \,,$$

where Ω is a domain of $\mathbb{R}^n$ with finite n-dimensional Lebesgue measure $|\Omega|$. Let R_1 be such that $|B_{R_1}| = 2|\Omega|$, where B_{R_1} is any ball of radius R_1 in $\mathbb{R}^n$. Let $Q_{R_1} = B_{R_1} \times (s - R_1^2, s)$, where s is any real number.

It is easy to check that $|Q_{R_1} \cap [\Omega \times (0,\infty)]| \leqq |Q_{R_1}|/2$ and hence

$$|Q_{R_1} \setminus D| \geqq |Q_{R_1} \setminus [\Omega \times (0,\infty)]| \geqq |Q_{R_1}|/2 \,.$$

We conclude that $R_1 = C(n)|\Omega|^{1/n}$, where $C(n)$ depends only on n, satisfies condition (PG) relative to $1/2, 1/2, D$.

(iii) Assume that

$$D \subset \mathbb{R}^n \times (0,T) \,.$$

Let R_2 be such that $R_2^2 = 2T$. Let $Q_{R_2} = B_{R_2} \times (s - R_2^2, s)$, where B_{R_2} is any ball of radius R_2 in $\mathbb{R}^n$ and s is any real number.

As in (ii), it is easy to check that $|Q_{R_2} \cap [\mathbb{R}^n \times (0,T)]| \leqq |Q_{R_2}|/2$ and hence that $R_2 = (2T)^{1/2}$ satisfies condition (PG) relative to $1/2, 1/2, D$.

We will prove the following improvement of the ABP-Krylov-Tso estimate (Theorem 1.12). Here $d^{\,n/(n+1)}$ is replaced by $R^{\,n/(n+1)}$, where R is as in Definition 1.13. We recall that D may be either bounded or unbounded.

THEOREM 1.15. *Assume that R satisfies condition* (PG) *relative to σ, τ, D (see Definition* 1.13) *and $R \leqq R_0$, for some constants σ, τ, and R_0. Assume also* (1.16)

and that w is bounded above in D. Then

$$\sup_D w \leqq \sup_{\partial_p D} w^+ + C\, R^{n/(n+1)}\, ||f||_{L^{n+1}(D)}\,, \tag{1.19}$$

where C is a constant depending only on $n, c_0, C_0, bR_0, \sigma$, and τ.

As an immediate consequence of Theorem 1.15 and Remark 1.14 we conclude the following.

COROLLARY 1.16. *Assume* (1.16) *and that w is bounded above in D.*

(i) *If D has finite $(n+1)$-dimensional Lebesgue measure $|D|$ then*

$$\sup_D w \leqq \sup_{\partial_p D} w^+ + C\, |D|^{\,n/((n+1)(n+2))}\, ||f||_{L^{n+1}(D)}\,, \tag{1.20}$$

where C is a constant depending only on n, c_0, C_0, and $b|D|^{1/(n+2)}$.

(ii) *If $D \subset \Omega \times (0,\infty)$ and Ω has finite n-dimensional Lebesgue measure $|\Omega|$ then*

$$\sup_D w \leqq \sup_{\partial_p D} w^+ + C\, |\Omega|^{1/(n+1)}\, ||f||_{L^{n+1}(D)}\,, \tag{1.21}$$

where C is a constant depending only on n, c_0, C_0, and $b|\Omega|^{1/n}$.

(iii) *If $D \subset \mathbb{R}^n \times (0,T)$ then*

$$\sup_D w \leqq \sup_{\partial_p D} w^+ + C\, T^{\,n/(2(n+1))}\, ||f||_{L^{n+1}(D)}\,, \tag{1.22}$$

where C is a constant depending only on n, c_0, C_0, and $bT^{1/2}$.

Note that in the case of a cylindrical domain $D = \Omega \times (0,T)$, (1.20) is an interpolation of (1.21) and (1.22).

Part (c). Plan of the Paper

In Section 2 we prove Theorem 1.4, Corollary 1.7, and give examples of domains in which Theorem 1.4 applies. In Section 3 we prove Theorem 1.8. In Section 4 we present a new statement and a new proof of the reversed Hölder inequality for the Green's function. In Section 5 we prove Theorems 1.9 and 1.11. Section 6 is devoted to the parabolic case.

2. Improved ABP Estimate and an Application

Let us recall condition (G) from Definition 1.2 and note that if $R \leqq R'$ and R satisfies condition (G) relative to σ, τ, Ω, then R' does as well. If $\tilde{\Omega} \subset \Omega$ and R satisfies condition (G) relative to σ, τ, Ω, then R satisfies condition (G) relative to $\sigma, \tau, \tilde{\Omega}$.

We point out that a ball B_{R_x} may satisfy (1.5) and $|B_{R_x} \setminus \Omega| = 0$ at the same time. The following is an easy example of this. Take, in $\mathbb{R}^2$,

$$\Omega = B_2(0) \setminus \{(x_1, x_2) \ : \ -1 \leqq x_1 \leqq 2, x_2 = 0\}$$

and $B_{1/2}(0)$ as the ball B_{R_x}. Note that $B_{2R_x} \cap \Omega = B_1(0) \cap \Omega$ has two components, and hence (1.5) holds for $\sigma = \tau = 1/2$ and any $x \in B_{1/2}(0) \cap \Omega$.

The previous example shows that inward pointing spikes removed from a domain in $\mathbb{R}^2$ (or inward boundary cusps) may have the effect that smaller numbers R will satisfy condition (G). In the case of a domain in $\mathbb{R}^n$, inward "pieces" of $(n-1)$-dimensional surfaces removed from the domain (or inward boundary cusps) may have the same effect (the reason being again that they can separate balls).

Remark 2.1. We give some examples of domains for which there is some R satisfying condition (G).

(i) Consider $\mathbb{R}^n$ as $\mathbb{R}^{n-k} \times \mathbb{R}^k$, $0 < k < n$, and let Ω be a domain of $\mathbb{R}^n$ such that for any $y \in \mathbb{R}^{n-k}$

$$|\{z \in \mathbb{R}^k \ : \ (y, z) \in \Omega\}| \leqq h\,,$$

where $|\cdot| = |\cdot|_k$ denotes the k-dimensional Lebesgue measure and h is a constant. Then, for a constant $C(n, k)$ depending only on n and k,

$$R = C(n, k)\, h^{1/k}$$

satisfies condition (G) relative to $1/2, 1/2, \Omega$.

This is easily seen as follows. Take R such that $|B_R^{(n)}|_n = 2h|B_R^{(n-k)}|_{n-k}$, where $B_R^{(s)}$ denotes a ball of radius R in $\mathbb{R}^s$. Apply Fubini's theorem to get, for any ball $B_R^{(n)}$,

$$\begin{aligned} 2|B_R^{(n)} \cap \Omega|_n &\leqq 2\int_{B_R^{(n-k)}} |\{z \in \mathbb{R}^k \ : \ (y, z) \in \Omega\}|_k \, dy \\ &\leqq 2h|B_R^{(n-k)}|_{n-k} \quad = \quad |B_R^{(n)}|_n\,. \end{aligned}$$

It follows that $|B_R^{(n)} \setminus \Omega| \geqq |B_R^{(n)}|/2$.

(ii) Assume that Ω is a domain such that for any $r > 0$

$$m_r(\Omega \cap \partial B_r) \leqq l\,,$$

where m_r is the volume measure on $\partial B_r = \partial B_r(0)$ (normalized such that $m_r(\partial B_r) = m_1(\partial B_1) r^{n-1}$) and l is a constant. Then, for a constant $C(n)$ depending only on n,

$$R = C(n)\, l^{1/(n-1)}$$

satisfies condition (G) relative to $1/2, 1/2, \Omega$.

To see this, take R such that $|B_R| = 4\,lR$ and use polar coordinates to check that, for any ball $B_R(x_0)$,

$$2|B_R(x_0) \cap \Omega| \leqq 2 \int_{\max(0,|x_0|-R)}^{|x_0|+R} dr \int_{\Omega \cap \partial B_r} dm_r \leqq 4\,lR = |B_R(x_0)| \,.$$

(iii) Assume that Ω is contained in, say,

$$\mathbb{R}^n \setminus \bigcup_{p \in \mathbb{Z}^n} B_{1/10}(p) \,,$$

where $\mathbb{Z}$ denotes the integer numbers. Then it is clear that there exists some R satisfying condition (G) relative to $\sigma, 1/2, \Omega$, for some σ.

The following result is the Krylov-Safonov boundary weak Harnack inequality due to Trudinger; see Theorem 9.27 in [6]. It will be the only strong tool used in the proof of Theorem 1.4.

THEOREM 2.2. (TRUDINGER) *Assume that $c \in L^\infty(\Omega)$ and $|c| \leqq \tilde{b}$ in Ω, for a positive constant $\tilde{b}$ (here we make no assumption on the sign of c). Let $0 < \tau < 1$ be a constant and B_R be a ball of radius R in $\mathbb{R}^n$ such that $B_R \cap \Omega$ and $B_{R/\tau} \setminus \Omega$ are nonempty. Suppose $R \leqq R_0$, for some positive constant R_0. Let $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ and $f \in L^n(\Omega)$ satisfy*

$$Lu \leqq f \quad \text{in } \Omega \quad , \quad u \geqq 0 \quad \text{in } \Omega \,.$$

Set

$$s = \liminf_{x \to B_{R/\tau} \cap \partial\Omega} u(x)$$

and

$$u_s^-(x) = \begin{cases} \inf\{u(x), s\} & \text{if } x \in \Omega \\ s & \text{if } x \notin \Omega \,. \end{cases}$$

Then

$$\left(\frac{1}{|B_R|} \int_{B_R} (u_s^-)_+^p \right)^{1/p} \leqq C \left(\inf_{B_R \cap \Omega} u + R \, ||f||_{L^n(B_{R/\tau} \cap \Omega)} \right), \tag{2.1}$$

where p and C are positive constants depending only on $n, c_0, C_0, bR_0, \tilde{b}R_0^2$, and τ.

Theorem 2.2 is stated in [6] for $\tau = 1/2$. It also holds for any $0 < \tau < 1$, as pointed out in the proofs of Theorems 9.27 and 9.22 in [6].

Before proving Theorem 1.4 let us recall that functions in $W^{2,n}_{\mathrm{loc}}(\Omega)$ are continuous in Ω. Hence, if Ω is bounded, $w \in W^{2,n}_{\mathrm{loc}}(\Omega)$, and $\limsup_{x \to \partial\Omega} w^+(x) < \infty$

then w is bounded above in Ω; this is not true if Ω is unbounded. The following simple example shows that the assumption w is bounded above in Ω is needed (in Theorem 1.4) in the case of an unbounded domain. Consider $w(x_1, x_2) = e^{x_1}\cos x_2$ in $\Omega = (-\infty, \infty) \times (-\pi/2, \pi/2)$; it satisfies $\Delta w = 0$ in Ω and $w = 0$ on $\partial\Omega$.

We can now give the

Proof of Theorem 1.4: We first make a standard reduction to the case $c \equiv 0$, in the following way. Consider

$$\tilde{w} = w - \limsup_{x \to \partial\Omega} w^+(x) .$$

Since $W^{2,n}_{\mathrm{loc}}(\Omega) \subset C(\Omega)$, the set $\tilde{\Omega} = \{x \in \Omega : \tilde{w}(x) > 0\}$ and its components are open. Note that $\limsup_{x \to \partial\tilde{\Omega}} \tilde{w}(x) \leqq 0$ and, recalling (1.1) for the definition of M and that $c \leqq 0$,

$$M\tilde{w} \geqq f - cw \geqq f - c\tilde{w} \geqq f \quad \text{in } \tilde{\Omega} .$$

Since $\tilde{\Omega} \subset \Omega$, we have that R also satisfies condition (G) relative to σ, τ, H, for any component H of $\tilde{\Omega}$.

We conclude that it is enough to prove

$$\sup_{\Omega} w \leqq CR\, ||f||_{L^n(\Omega)} \tag{2.2}$$

when $Mw \geqq f$ in Ω, $\limsup_{x \to \partial\Omega} w(x) \leqq 0$, and $0 < \sup_{\Omega} w < \infty$.

To prove this, assume first that Ω is bounded. We take $y \in \Omega$ such that

$$K := \sup_{\Omega} w = w(y) > 0$$

and, since R satisfies condition (G) relative to σ, τ, Ω, a ball B_{R_y} such that

$$R_y \leqq R \quad , \quad y \in B_{R_y} \quad \text{and} \quad |B_{R_y} \setminus \Omega_y| \geqq \sigma |B_{R_y}| . \tag{2.3}$$

Recall that Ω_y is the component of $B_{R_y/\tau} \cap \Omega$ containing y.

Consider the function

$$u = K - w .$$

It satisfies $Mu \leqq -f$ in Ω and $u \geqq 0$ in Ω. Note also that

$$y \in B_{R_y} \cap \Omega_y \quad \text{and} \quad u(y) = 0 . \tag{2.4}$$

We now apply the boundary weak Harnack inequality (Theorem 2.2) to u, the operator M in Ω_y and the ball B_{R_y}. Note that $B_{R_y/\tau} \setminus \Omega_y \neq \varnothing$ by (2.3), and that

$$B_{R_y/\tau} \cap \partial\Omega_y \subset B_{R_y/\tau} \cap \partial(B_{R_y/\tau} \cap \Omega) \subset B_{R_y/\tau} \cap \partial\Omega .$$

Since $\limsup_{x \to \partial\Omega} w(x) \leqq 0$ we have, with the notation of Theorem 2.2,

$$s = \liminf_{x \to B_{R_y/\tau} \cap \partial\Omega_y} u(x) \geqq K \quad , \quad u_s^- \geqq K \text{ in } B_{R_y} \setminus \Omega_y . \tag{2.5}$$

Theorem 2.2 gives

$$\left(\frac{1}{|B_{R_y}|}\int_{B_{R_y}}(u_s^-)^p\right)^{1/p} \leqq C\left(\inf_{B_{R_y}\cap\Omega_y} u + R_y\,||f||_{L^n(\Omega_y)}\right).$$

This implies, with the aid of (2.3), (2.4), and (2.5),

$$\begin{aligned}\sigma^{1/p}K &\leqq \left(\frac{1}{|B_{R_y}|}\int_{B_{R_y}\setminus\Omega_y}(u_s^-)^p\right)^{1/p} \leqq \left(\frac{1}{|B_{R_y}|}\int_{B_{R_y}}(u_s^-)^p\right)^{1/p}\\ &\leqq CR_y\,||f||_{L^n(B_{R_y/\tau}\cap\Omega)} \leqq CR\,||f||_{L^n(\Omega)}\end{aligned} \tag{2.6}$$

where p and C are positive constants depending only on n, c_0, C_0, bR_0, and τ. Estimate (2.2) is now proved.

In case that Ω is unbounded, the proof is the same with minor changes; we define $K := \sup_\Omega w$ and we now take, for any $\eta > 0$, a point $y \in \Omega$ such that $K - \eta \leqq w(y)$. We now have that $u(y) \leqq \eta$. The proof proceeds in the same way as before and we get the desired estimate by letting $\eta \to 0$ at the end of the proof.

Remark 2.3. Under the hypothesis of the improved ABP estimate (Theorem 1.4), we have that

$$\sup_\Omega w \leqq \limsup_{x\to\partial\Omega} w^+(x) + CR^2\,||f||_{L^\infty(\Omega)}\,, \tag{2.7}$$

with C as in Theorem 1.4.

This follows from the proof of Theorem 1.4 in the following way; when Ω is bounded, (2.6) implies that $||f||_{L^n(\Omega)}$ in the improved ABP estimate (1.7) may be replaced by $||f||_{L^n(B_{R/\tau}\cap\Omega)}$, for some ball $B_{R/\tau}$ of radius R/τ (note that this ball depends on w). Using that $||f||_{L^n(B_{R/\tau}\cap\Omega)} \leqq C(n)\tau^{-1}R\,||f||_{L^\infty(\Omega)}$, we get (2.7). In case that Ω is unbounded, take η (see the final part of the proof of Theorem 1.4) such that $C\eta = \sigma^{1/p}K/2$, where C and p are as in (2.6).

We use the previous remark to prove Corollary 1.7.

Proof of Corollary 1.7: Let w be bounded above satisfy $\limsup_{x\to\partial\Omega} w(x) \leqq 0$ and $Lw \geqq 0$ in Ω. Hence $(M - c^-)w \geqq -c^+w \geqq -c^+w^+$ and we can apply (2.7) to $M - c^-$ and w. We get

$$\sup_\Omega w \leqq C(R^*)^2\,||c^+w^+||_{L^\infty(\Omega)} \leqq C(R^*)^2 b' \sup_\Omega w^+\,,$$

where C depends only on n, c_0, C_0, b, and σ (we take $R^* \leqq 1$). Taking R^* small enough such that $C(R^*)^2 b' \leqq 1/2$, we conclude that $w \leqq 0$ in Ω.

Remark 2.4. As a consequence of Corollary 1.7 we have that the maximum principle for L in Ω (we assume here that $c \leqq b'$ in Ω) holds if one of the following quantities is small enough depending only on n, c_0, C_0, b, and b':

(a) $|\Omega|$ (see Remark 1.3).
(b) h, as in (i) of Remark 2.1.
(c) l, as in (ii) of Remark 2.1.

3. Proof of Theorem 1.8

We start this section giving some examples of domains for which Theorem 1.8 applies. Assume that $0 < \sigma < 1$ and Ω is a domain such that, for any $r > R_1$,

$$m_r(\Omega \cap \partial B_r) \leqq (1 - \sigma)\, m_r(\partial B_r)\,,$$

where m_r is the volume measure on $\partial B_r = \partial B_r(0)$ as in (ii) of Remark 2.1. Then condition (1.9) is obviously satisfied.

It follows that Theorem 1.8 applies in any domain Ω contained in a cone $\{x \in \mathbb{R}^n \setminus \{0\} \ : \ x/|x| \in \Gamma\}$, where Γ is any subset of the unit sphere with $m_1(\partial B_1 \setminus \Gamma) > 0$. In this case we can take $R_1 = 0$.

In order to prove Theorem 1.8, we will need the following result. It states that the boundary weak Harnack inequality (Theorem 2.2) also holds when the balls B_R and $B_{R/\tau}$ are replaced by annuli $A(R, 2R)$ and $A(R/2, 4R)$ respectively.

THEOREM 3.1. *Assume that $c \in L^\infty(\Omega)$ and $|c| \leqq \tilde{b}$ in Ω, for a positive constant $\tilde{b}$. Let $H = A(R, 2R)$ and $\tilde{H} = A(R/2, 4R)$ be annuli such that $H \cap \Omega$ and $\tilde{H} \setminus \Omega$ are nonempty. Suppose $R \leqq R_0$, for some positive constant R_0. Let $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ and $f \in L^n(\Omega)$ satisfy*

$$Lu \leqq f \quad \text{in } \Omega \quad , \quad u \geqq 0 \quad \text{in } \Omega\,.$$

Set

$$s = \liminf_{x \to \tilde{H} \cap \partial\Omega} u(x)$$

and

$$u_s^-(x) = \begin{cases} \inf\{u(x), s\} & \text{if } x \in \Omega \\ s & \text{if } x \notin \Omega\,. \end{cases}$$

Then

$$\left(\frac{1}{|H|} \int_H (u_s^-)^p \right)^{1/p} \leqq C \left(\inf_{H \cap \Omega} u + R\, ||f||_{L^n(\tilde{H} \cap \Omega)} \right), \tag{3.1}$$

where p and C are positive constants depending only on n, c_0, C_0, bR_0, and $\tilde{b}R_0^2$.

Remark 3.2. Theorem 3.1 also holds for more general sets than annuli. More precisely, let H and $\tilde{H}$ be open sets for which there exist N balls $B^1_\rho, \ldots, B^N_\rho$ of radius $\rho \leqq \rho_0$ (ρ_0 is here a fixed constant) such that

$$H \subset B^1_\rho \cup \ldots \cup B^N_\rho \qquad , \qquad B^1_{2\rho} \cup \ldots \cup B^N_{2\rho} \subset \tilde{H}\,,$$

$$|H| \geqq \mu\rho^n \qquad \text{and} \qquad |B_\rho^i \cap B_\rho^{i+1}| \geqq \mu\rho^n \qquad \forall\ 1 \leqq i \leqq N-1\,,$$

for some constant $\mu > 0$. Then Theorem 3.1, with R replaced by ρ in (3.1), holds for H and $\tilde{H}$. Here the constants p and C in (3.1) depend only on $n, c_0, C_0, b\rho_0, \tilde{b}\rho_0^2$, μ, and N.

Note that if $H = A(R, 2R)$ and $\tilde{H} = A(R/2, 4R)$ then there exist balls such that the conditions for H and $\tilde{H}$ in the previous remark are satisfied. We can take $\rho = R/4$ and N and μ depending only on n.

We now prove Theorem 3.1 for the case of two open sets H and $\tilde{H}$ as in the previous remark.

Proof of Theorem 3.1: We will prove that for any $1 \leqq i \leqq N$ and $1 \leqq j \leqq N-1$,

$$\left(\frac{1}{|B_\rho^i|}\int_{B_\rho^i}(u_s^-)^p\right)^{1/p} \leqq C\left(\inf_{B_\rho^i} u_s^- + \rho\,||f||_{L^n(B_{2\rho}^i\cap\Omega)}\right) \tag{3.2}$$

and

$$\inf_{B_\rho^j} u_s^- \leqq C\left(\frac{1}{|B_\rho^{j+1}|}\int_{B_\rho^{j+1}}(u_s^-)^p\right)^{1/p}, \tag{3.3}$$

for some positive constants p and C which depend — as all C's in this proof — on the quantities stated in the previous remark.

Since $H \subset B_\rho^1 \cup \ldots \cup B_\rho^N$ and $|H| \geqq \mu\rho^n$, it follows that

$$\left(\frac{1}{|H|}\int_H (u_s^-)^p\right)^{1/p} \leqq C\left(\frac{1}{|B_\rho^{i_1}|}\int_{B_\rho^{i_1}}(u_s^-)^p\right)^{1/p}$$

and

$$\inf_{B_\rho^{i_2}} u_s^- \leqq \inf_H u_s^- \leqq \inf_{H\cap\Omega} u\,,$$

for some $i_1 \leqq i_2$ (relabel the balls B_ρ^i, if necessary, to have $i_1 \leqq i_2$). These two inequalities combined with (3.2) and (3.3) — used repeatedly — give (3.1) with R replaced by ρ.

It remains to check (3.2) and (3.3). Since $|B_\rho^j \cap B_\rho^{j+1}| \geqq \mu\rho^n$, we have that

$$\begin{aligned}\inf_{B_\rho^j} u_s^- &\leqq \inf_{B_\rho^j\cap B_\rho^{j+1}} u_s^- \leqq \left(\frac{1}{|B_\rho^j \cap B_\rho^{j+1}|}\int_{B_\rho^j\cap B_\rho^{j+1}}(u_s^-)^p\right)^{1/p}\\ &\leqq C\left(\frac{1}{|B_\rho^{j+1}|}\int_{B_\rho^{j+1}}(u_s^-)^p\right)^{1/p}\end{aligned}$$

and we conclude (3.3).

To prove (3.2), first suppose that $B^i_{2\rho} \setminus \Omega \neq \emptyset$. We can assume that $B^i_\rho \cap \Omega \neq \emptyset$, since otherwise $u_s^- \equiv s$ in B^i_ρ and (3.2) becomes trivial. We apply the boundary weak Harnack inequality (Theorem 2.2) in B^i_ρ with $\tau = 1/2$. We get $s_i = \liminf_{x \to B^i_{2\rho} \cap \partial\Omega} u \geqq \liminf_{x \to \bar{H} \cap \partial\Omega} u = s$, $u_{s_i}^- \geqq u_s^-$ in $B^i_{2\rho}$ and

$$\begin{aligned}\left(\frac{1}{|B^i_\rho|}\int_{B^i_\rho}(u_s^-)^p\right)^{1/p} &\leqq \left(\frac{1}{|B^i_\rho|}\int_{B^i_\rho}(u_{s_i}^-)^p\right)^{1/p} \\ &\leqq C\left(\inf_{B^i_\rho \cap \Omega} u + \rho\,||f||_{L^n(B^i_{2\rho}\cap\Omega)}\right).\end{aligned}$$

Since $u_s^- \leqq s$, we conclude that

$$\left(\frac{1}{|B^i_\rho|}\int_{B^i_\rho}(u_s^-)^p\right)^{1/p} \leqq (C+1)\left(\inf_{B^i_\rho} u_s^- + \rho\,||f||_{L^n(B^i_{2\rho}\cap\Omega)}\right),$$

that is (3.2).

We finally assume $B^i_{2\rho} \subset \Omega$. We now apply the weak Harnack inequality (Theorem 9.22 in [6]) to get

$$\left(\frac{1}{|B^i_\rho|}\int_{B^i_\rho} u^p\right)^{1/p} \leqq C\left(\inf_{B^i_\rho} u + \rho\,||f||_{L^n(B^i_{2\rho})}\right),$$

and conclude (3.2) as before. Theorem 3.1 is now proved.

We point out that the previous covering argument is already used in the proof given in [6] of the weak Harnack inequality (Theorem 9.22 in [6]). We can now give the

Proof of Theorem 1.8: We proceed as in the proof of the improved ABP estimate (Theorem 1.4). We reduce the problem to prove

$$K := \sup_\Omega w \leqq C\,||(R_1 + |x|)f(x)||_{L^n(\Omega)}, \tag{3.4}$$

when $Mw \geqq f$ in Ω, $\limsup_{x\to\partial\Omega} w(x) \leqq 0$ and $K > 0$.

For any $\eta > 0$ we take a point $0 \neq y \in \Omega$ such that $K - \eta \leqq w(y)$.

Suppose first that $|y| \leqq R_1$. Then $R_1 > 0$ and, by (1.9),

$$|B_{2R_1}(0) \setminus \Omega| \geqq |A(R_1, 2R_1) \setminus \Omega| \geqq \sigma|A(R_1, 2R_1)| = C(n,\sigma)|B_{2R_1}(0)|,$$

for a positive constant $C(n,\sigma)$ depending only on n and σ. As in the proof of Theorem 1.4, we now apply the boundary weak Harnack inequality (Theorem 2.2), with $\tau = 1/2$, to $u = K - w$, the operator M and the ball $B_{2R_1}(0)$. Note that $y \in B_{2R_1}(0)$ and $u(y) \leqq \eta$. Recalling that $R_1\,(\sum b_i(x)^2)^{1/2} \leqq B$ for any $x \in \Omega$, we conclude that

$$K \leqq C(\eta + R_1\,||f(x)||_{L^n(\Omega)}),$$

where C is a constant as in (1.10).

Suppose now that $|y| > R_1$. Define $R = |y|$, $H = A(R, 2R)$, and $\tilde{H} = A(R/2, 4R)$. We can then apply Theorem 3.1 to $u = K - w$, the operator M in Ω, H, and $\tilde{H}$. Proceeding as in the proof of Theorem 1.4 and using (1.9) (with $r = R$) and (3.1), we conclude (note that $y \in \overline{H}$)

$$\sigma^{1/p} K \leqq C\left(\eta + R\,||f||_{L^n(\tilde{H}\cap\Omega)}\right),$$

where p and C are positive constants depending only on n, c_0, C_0, and

$$R \sup_{\tilde{H}\cap\Omega} (\textstyle\sum b_i^2)^{1/2}.$$

But note that $R \leqq 2|x|$ for any $x \in \tilde{H} = A(R/2, 4R)$. It follows

$$K \leqq C\left(\eta + ||\,|x| f(x)\,||_{L^n(\Omega)}\right),$$

where C depends only on n, c_0, C_0, B, and σ.

We conclude that we always have $K \leqq C(\eta + ||(R_1 + |x|)f(x)||_{L^n(\Omega)})$. Letting $\eta \to 0$, we conclude (3.4).

Remark 3.3. Set $\vec{b} = (b_1, \ldots, b_n)$. The ABP estimate (Theorem 1.1) holds if $\vec{b} \in L^n(\Omega)$ (see Theorem 9.1 in [6]). The boundary weak Harnack inequality (Theorem 2.2) holds if $\vec{b} \in L^{2n}(\Omega)$ and $c \in L^n(\Omega)$; see [11] and Notes to Chapter 9 in [6].

Since we applied Theorem 2.2 to the operator M, the improved ABP estimate (Theorem 1.4) and hence Corollary 1.5 hold if $\vec{b} \in L^{2n}(\Omega)$. It also follows that Theorem 1.8 holds if $|x|^{1/2}\vec{b}(x) \in L^{2n}(\Omega)$.

4. A New Proof of the Reversed Hölder Inequality for the Green's Function

In this section we deal with a bounded domain Ω and we make the following regularity assumption. It will be just a qualitative assumption, since the estimates will not depend on the modulus of regularity assumed.

ASSUMPTION (S): *$n \geqq 2$, Ω is bounded, $\partial\Omega$ is C^∞, the coefficients of M are defined and are C^∞ in $\mathbb{R}^n$, and both* (1.2) *and* (1.3) *hold for all $x \in \mathbb{R}^n$.*

We recall that, under Assumption (S), the Dirichlet problem for M in Ω is well posed in suitable spaces; see Chapter 9 in [6]. We point out that, from now on, by smooth we mean of class C^∞.

The following result is the Fabes and Stroock reversed Hölder inequality for the Green's function corresponding to the operator M and Ω. It is proved in [4] for operators containing only second-order terms (see Theorem 2.2 in [4]), but their

proof also applies to our operator M. In this section we give a new proof of the inequality for the operator M.

THEOREM 4.1. (FABES, STROOCK) *Assume* (S) *and let R_0 be a positive constant and $G(x,y)$ be the Green's function corresponding to M and Ω. Then, for any ball B_R of radius R in $\mathbb{R}^n$ such that $R \leqq R_0$ and $B_{4R} \subset \Omega$,*

$$\left(\frac{1}{R^n}\int_{B_R} G(x,y)^{n/(n-1)}\,dy\right)^{(n-1)/n} \leqq C\,\frac{1}{R^n}\int_{B_R} G(x,y)\,dy \qquad \forall\, x \in \Omega\,,$$

where C is a positive constant depending only on n, c_0, C_0, and bR_0.

Note that, for $h \in L^n(\Omega)$, the solution of the boundary value problem

$$\begin{cases} M\tilde{w} &= -h \quad \text{in } \Omega \\ \tilde{w} &= 0 \quad \text{on } \partial\Omega \end{cases}$$

is given by $\tilde{w}(x) = \int_\Omega G(x,y)h(y)\,dy$, for any $x \in \Omega$, where $G(x,y)$ is the Green's function corresponding to M and Ω. It follows that Theorem 4.1 is equivalent to the following.

THEOREM 4.2. *Assume* (S) *and let R_0 be a positive constant and B_R be a ball of radius R in $\mathbb{R}^n$ such that $R \leqq R_0$ and $B_{4R} \subset \Omega$. For $f \in L^n(\Omega)$ with support contained in $\overline{B}_R$, let u and v be the solutions of*

$$\begin{cases} Mu &= -f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \qquad \begin{cases} Mv &= -\chi_{B_R} \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega\,, \end{cases}$$

where χ_{B_R} is the characteristic function of B_R. Then

$$u(x) \leqq C\,\frac{\|f\|_{L^n(\Omega)}}{R}\,v(x) \qquad \forall\, x \in \Omega\,, \tag{4.1}$$

where C is a constant depending only on n, c_0, C_0, and bR_0.

Throughout this section C (and $C_1, C_2, \ldots$) will denote *positive* constants depending only on n, c_0, C_0, and bR_0. Each one of them may be different from formula to formula.

Let us introduce two comparison functions that will be used in the proof of Theorem 4.2. We may assume that $B_R = B_R(0)$. Consider, for $k \geqq 1$ integer and $0 < \delta < 1$, the functions

$$\phi_1(x) = \phi_1(|x|) = R^{2-2k}[(1+\delta)^{2k}R^{2k} - |x|^{2k}] \qquad \text{and}$$

$$\phi_2(x) = \phi_2(|x|) = \delta R^{-2}[(1+\delta)^2R^2 - |x|^2]^2\,.$$

It is easy to verify that, for $R \leqq R_0$, there exist constants (that we fix for the rest of this section) k sufficiently large and δ sufficiently small, both depending only on n, c_0, C_0, and bR_0, such that

$$M\phi_1 \leqq 0 \quad \text{for } |x| \leqq (1+\delta)R = R'' ,$$

$$M\phi_2 \geqq -1 \quad \text{for } |x| \leqq R \qquad \text{and}$$

$$M\phi_2 \geqq 0 \quad \text{for } R \leqq |x| \leqq (1+\delta)R = R'' ;$$

we define, to simplify notation,

$$R' = (1 + (\delta/2))R \qquad \text{and} \qquad R'' = (1+\delta)R .$$

We will also need that, for $i = 1, 2$ (C_i may be different in each formula)

$$\phi_i(R') = C_i R^2 \qquad , \qquad \partial\phi_i/\partial|x|\,(R') = -C_i R$$

$$\text{and} \qquad \phi_i(x) - \phi_i(R') \geqq C_i R^2 \quad \text{for } |x| \leqq R .$$

The other tool that we use is the Krylov-Safonov Harnack inequality; see Corollary 9.25 in [6]. It will be used only in the proof of the following lemma towards Theorem 4.2.

LEMMA 4.3. *Under the assumptions of Theorem* 4.2 *and with R' as above, let w be the solution of*

$$\begin{cases} Mw &= 0 \quad \text{in } \Omega \setminus \overline{B}_{R'} \\ w &= 1 \quad \text{on } \partial B_{R'} \\ w &= 0 \quad \text{on } \partial\Omega . \end{cases}$$

Then

$$0 < \sup_{\partial B_{R'}} w_\nu \leqq C \inf_{\partial B_{R'}} w_\nu ,$$

where w_ν denotes the derivative of w with respect to the interior unit normal to $B_{R'}$ on $\partial B_{R'}$, and C is a constant depending only on n, c_0, C_0, and bR_0.

Proof: We have that $0 \leqq w \leqq 1$, by the maximum principle. Let us consider

$$\underline{m} = \inf_{\partial B_{R''}} w \qquad \text{and} \qquad \overline{m} = \sup_{\partial B_{R''}} w .$$

We apply the Harnack inequality (see Corollary 9.25 in [6]) to $1 - w$, combined with a covering argument over $\partial B_{R''}$, to get

$$1 - \underline{m} \leqq C(1 - \overline{m}) . \tag{4.2}$$

We now apply the maximum principle in $\{R' < |x| < R''\}$ to obtain (note that $\phi_i(R'') = 0$, for $i = 1, 2$)

$$(1 - \underline{m})\frac{\phi_2(x)}{\phi_2(R')} + \underline{m} \leqq w(x) \leqq (1 - \overline{m})\frac{\phi_1(x)}{\phi_1(R')} + \overline{m} \tag{4.3}$$

for $R' \leqq |x| \leqq R''$. Since (4.3) consists of equalities for $|x| = R'$, we conclude that, for $|x| = R'$,

$$(1-\overline{m})\frac{(\phi_1)_\nu(x)}{\phi_1(R')} \leqq w_\nu(x) \leqq (1-\underline{m})\frac{(\phi_2)_\nu(x)}{\phi_2(R')}\,.$$

Therefore we get, by (4.2),

$$(1-\overline{m})C_1R^{-1} \leqq w_\nu(x) \leqq (1-\underline{m})C_2R^{-1} \leqq (1-\overline{m})C_3R^{-1}\,,$$

that proves the lemma.

Proof of Theorem 4.2: By an approximation argument, we may assume that $f \in L^\infty(\Omega)$. We then have that u and v belong to $W^{2,p}(\Omega)$, for any $p > n$, and hence to $C^1(\Omega)$.

Let $x_1 \in \partial B_{R'}$ and $x_2 \in \partial B_{R'}$ be such that

$$u(x_1) = \sup_{\partial B_{R'}} u \qquad \text{and} \qquad v(x_2) = \inf_{\partial B_{R'}} v\,.$$

We consider the function w as in Lemma 4.3 and apply the maximum principle in $\Omega \setminus \overline{B}_{R'}$, to get $v \geqq v(x_2)w$ in $\Omega \setminus \overline{B}_{R'}$. Since there is equality at x_2, we obtain

$$v_\nu(x_2) \leqq v(x_2)w_\nu(x_2)\,. \tag{4.4}$$

Similarly, we have (recall that f has support in $\overline{B}_R$)

$$u_\nu(x_1) \geqq u(x_1)w_\nu(x_1)\,. \tag{4.5}$$

Note that $\phi_2 - \phi_2(R') \leqq v - v(x_2)$ in $\overline{B}_{R'}$, by the maximum principle in $B_{R'}$. It follows (recall that $v \geqq 0$ in Ω) that

$$v \geqq CR^2 \quad \text{in } \overline{B}_R\,, \tag{4.6}$$

$$CR \leqq v_\nu(x_2)\,. \tag{4.7}$$

We now apply the ABP estimate in $B_{R'}$ and obtain (note that $u - u(x_1) \leqq 0$ on $\partial B_{R'}$)

$$u - u(x_1) \leqq CR\,||f||_{L^n(\Omega)} = CR^{-1}||f||_{L^n(\Omega)}\,R^2 \quad \text{in } \overline{B}_R\,. \tag{4.8}$$

Hence $u - u(x_1) \leqq CR^{-1}||f||_{L^n(\Omega)}\,[\phi_1 - \phi_1(R')]$ in $\overline{B}_R$ and, by the maximum principle, in $\overline{B}_{R'}$. We conclude

$$u_\nu(x_1) \leqq CR^{-1}||f||_{L^n(\Omega)}\,R\,.$$

This combined with (4.5), (4.7), and (4.4) gives (recall that $w_\nu(x_1) > 0$ by Lemma 4.3)

$$\begin{aligned} u(x_1) &\leqq \frac{u_\nu(x_1)}{w_\nu(x_1)} \leqq CR^{-1}||f||_{L^n(\Omega)}\,\frac{R}{w_\nu(x_1)} \\ &\leqq CR^{-1}||f||_{L^n(\Omega)}\,\frac{v_\nu(x_2)}{w_\nu(x_1)} \leqq CR^{-1}||f||_{L^n(\Omega)}\,\frac{w_\nu(x_2)}{w_\nu(x_1)}\,v(x_2)\,. \end{aligned}$$

This implies, by Lemma 4.3,

$$u(x_1) \leqq CR^{-1}||f||_{L^n(\Omega)}\, v(x_2)\,. \tag{4.9}$$

Note that, by our choice of x_2, $v(x_2) \leqq v$ in $\overline{B}_{R'}$. This combined with (4.8), (4.9), and (4.6) gives

$$\begin{aligned} u &\leqq u(x_1) + CR^{-1}||f||_{L^n(\Omega)}\, R^2 \\ &\leqq CR^{-1}||f||_{L^n(\Omega)}\, v(x_2) + CR^{-1}||f||_{L^n(\Omega)}\, R^2 \\ &\leqq CR^{-1}||f||_{L^n(\Omega)}\, v \quad \text{in } \overline{B}_R\,, \end{aligned}$$

and hence in Ω, by the maximum principle. Estimate (4.1) and Theorem 4.2 are now proved.

SUGGESTION: The idea of this proof becomes more clear when adapting it to the case where all the objects are radially symmetric. Then the proof becomes very simple!

5. Some Applications of the Reversed Hölder Inequality for the Green's Function

In this section we prove Theorems 1.9 and 1.11. In [4] Fabes and Stroock use their reversed Hölder inequality and a result by Gehring to improve the ABP estimate. We state this result in Corollary 5.1 and, for the sake of completeness, we present its proof (as given in [4]) in an appendix.

COROLLARY 5.1. (FABES, STROOCK) *Assume* (S) (*see Section* 4) *and that* $\operatorname{diam}(\Omega) \leqq d$, *for a positive constant* d. *Let* $G(x,y)$ *be the Green's function corresponding to* M *and* Ω. *Then there exist constants* $p_0 \in (n/2, n)$ *and* $C > 0$, *depending only on* n, c_0, C_0, *and* bd, *such that*

$$\left(\int_\Omega G(x,y)^{q_0}\, dy\right)^{1/q_0} \leqq C \operatorname{diam}(\Omega)^{2-n/p_0} \qquad \forall\, x \in \Omega\,,$$

where q_0 *is the conjugate exponent of* p_0 (*and hence* $q_0 > n/(n-1)$).

Note that the ABP estimate implies that the above inequality holds with $q_0 = n/(n-1)$, which is the conjugate exponent of n.

Proof of Theorem 1.9: Since $p_0 > n/2$ we have that $W^{2,p_0}_{\mathrm{loc}}(\Omega) \subset C(\Omega)$. Therefore we can reduce our problem to the case $c \equiv 0$, as in the proof of Theorem 1.4, and conclude that it is enough to prove

$$\sup_\Omega w \leqq CR^{2-n/p_0}\, ||f||_{L^{p_0}(\Omega)} \tag{5.1}$$

when

$$(5.2)\qquad Mw \geqq f \text{ in } \Omega\,, \quad \limsup_{x\to\partial\Omega} w(x) \leqq 0\,, \quad \text{and} \quad 0 < \sup_{\Omega} w < \infty\,.$$

Case 1. We first assume that Ω is bounded.

Note now that, for any $\varepsilon > 0$, there exists an open set $H \subset \Omega$ such that $\overline{H} \subset \Omega$, ∂H is C^∞ and $w \leqq \varepsilon$ in $\Omega \setminus H$. Considering $w - \varepsilon$ and letting ε tend to zero, we conclude that it suffices to prove

$$(5.3)\qquad \sup_H w \leqq CR^{2-n/p_0}\, ||f||_{L^{p_0}(H)}$$

when $H \subset \Omega$ is open, $\overline{H} \subset \Omega$, ∂H is C^∞, $w \in W^{2,p_0}_{\mathrm{loc}}(\Omega)$, $f \in L^{p_0}(H)$, $Mw \geqq f$ in H, $w \leqq 0$ on ∂H, and $\sup_H w > 0$.

We can further assume that $w \in C^\infty(\overline{H})$, since $C^\infty(\overline{H})$ is dense in $W^{2,p_0}(H)$.

We now convolute a_{ij} and b_i with an approximation of the identity (check that the ellipticity constants do not change) and extend the new coefficients smoothly to $\mathbb{R}^n$ (use a partition of the unity). In this way we approximate the coefficients of M in $L^{p_0}(H)$ by smooth ones. Since $w \in C^\infty(\overline{H})$, we can pass to the limit in (5.3), and conclude that it is enough to prove (5.1) under Assumption (S) (see Section 4) and

$$(5.4)\qquad w \in C^\infty(\overline{\Omega})\,, \quad Mw \geqq f \text{ in } \Omega\,, \quad w \leqq 0 \text{ on } \partial\Omega\,, \quad \text{and} \quad \sup_{\Omega} w > 0\,.$$

We now assume (S) and (5.4); we need to show (5.1). Let us take $y \in \Omega$ such that

$$K := \sup_{\Omega} w = w(y) > 0$$

and, since R satisfies condition (G) relative to σ, τ, Ω, a ball $B_{R_y} = B_\rho$ (we set $R_y = \rho$ to simplify notation) such that

$$(5.5)\qquad y \in B_\rho \quad \text{and} \quad |B_\rho \setminus \Omega_y| \geqq \sigma |B_\rho|\,,$$

where Ω_y is the component of $B_{\rho/\tau} \cap \Omega$ containing y.

Consider the function

$$v = K - w\,.$$

We have that $v \geqq 0$ in $\overline{\Omega}$, $v \geqq K$ on $\partial\Omega$, and $v(y) = 0$. Since (S) holds and $w \in C^\infty(\overline{\Omega})$, we can solve, by standard existence theory (see Chapter 9 in [6]),

$$\begin{cases} Mv_1 &= -(Mw)^- \chi_{\Omega_y} & \text{in } B_{\rho/\tau} \\ v_1 &= 0 & \text{on } \partial B_{\rho/\tau}\,. \end{cases}$$

We have that $v_1 \in W^{2,n}(B_{\rho/\tau})$. Corollary 5.1 applied with $B_{\rho/\tau}$ in place of Ω gives, since $(Mw)^- \leqq |f|$ and $\rho = R_y \leqq R$,

$$0 \leqq v_1 \leqq C_1 R^{2-n/p_0}\, ||f||_{L^{p_0}(\Omega)} \quad \text{in } \overline{B}_{\rho/\tau}\,,$$

where p_0 depends only on n, c_0, C_0, and bR_0/τ, while C_1 depends only on n, c_0, C_0, bR_0, and τ.

It is easy to check that, for

$$u = v + v_1 ,$$

$u \geqq 0$ in Ω_y and

$$\begin{cases} Mu & \leqq & 0 & \text{in } \Omega_y \\ u & \geqq & 0 & \text{on } \partial B_{\rho/\tau} \cap \overline{\Omega} \\ u & \geqq & K & \text{on } B_{\rho/\tau} \cap \partial\Omega . \end{cases}$$

Recall that $u \in W^{2,n}(\Omega_y) \cap C(\overline{\Omega_y})$ and $\partial\Omega_y \subset \partial(B_{\rho/\tau} \cap \Omega) \subset (\partial B_{\rho/\tau} \cap \overline{\Omega}) \cup (B_{\rho/\tau} \cap \partial\Omega)$.

We have that $Mu \leqq 0$ in Ω_y and $u \geqq 0$ in Ω_y and we can apply the boundary weak Harnack inequality (Theorem 2.2) to u, the operator M in Ω_y, and the ball B_ρ. With the notation of Theorem 2.2, we have that

$$s = \liminf_{x \to B_{\rho/\tau} \cap \partial\Omega_y} u(x) \geqq K \quad \text{and} \quad u_s^- \geqq K \text{ in } B_\rho \setminus \Omega_y .$$

Theorem 2.2, (5.5), and $u(y) = v_1(y) \leqq C_1 R^{2-n/p_0} \, ||f||_{L^{p_0}(\Omega)}$ give

$$\sigma^{1/p} K \leqq \left(\frac{1}{|B_\rho|} \int_{B_\rho} (u_s^-)^p \right)^{1/p} \leqq C C_1 R^{2-n/p_0} \, ||f||_{L^{p_0}(\Omega)} ,$$

where p and C are positive constants depending only on n, c_0, C_0, bR_0, and τ. Estimate (5.1) is now proved.

Case 2. Assume finally that Ω is unbounded.

Let w satisfy (5.2). We first define $K := \sup_\Omega w$ and take, for any $\eta > 0$, a point $y \in \Omega$ such that $K - \eta \leqq w(y)$. We now have that $v(y) \leqq \eta$. We consider B_ρ as in Case 1. Note that in the rest of the proof (for Case 1), $\limsup_{x \to B_{\rho/\tau} \cap \partial\Omega} w \leqq 0$ — rather than $\limsup_{x \to \partial\Omega} w \leqq 0$ — is needed. We now take $\varepsilon > 0$ and approximate $w - \varepsilon$ and the coefficients of M in $B_{\rho/\tau} \cap \{w > \varepsilon/2\}$ — whose closure is compact and contained in Ω — by smooth functions. Since $w - \varepsilon = -\varepsilon/2$ on $B_{\rho/\tau} \cap \partial\{w > \varepsilon/2\}$, we can take the smooth function w_ε that approximates $w - \varepsilon$ to be nonpositive on $B_{\rho/\tau} \cap \partial\{w > \varepsilon/2\}$.

We proceed as in Case 1, now applied to w_ε. We get the desired estimate by letting $\varepsilon \to 0$ first, and then $\eta \to 0$.

Proof of Theorem 1.11: We first note that, by scaling, it is enough to show (1.12) with C depending only on $n, c_0, C_0, b, \tilde{b}$, and R_0. Using a covering argument, it is easy to check that it suffices to prove the theorem under the assumption that $R \leqq R_1$, where R_1 — that will be small enough and chosen later — depends only on n, c_0, C_0, b, and $\tilde{b}$.

We approximate $u+\varepsilon$, for any $\varepsilon > 0$, in $W^{2,p_0}(B_{3R/2})$ by smooth functions. We then approximate the coefficients of L in $L^{p_0}(B_{3R/2})$ by smooth ones — as in the proof of Theorem 1.9. Letting ε tend to 0, we see that it is enough to prove

$$\sup_{B_R} u \leqq C\left(\inf_{B_R} u + R^{2-n/p_o}\,||f||_{L^{p_0}(B_{3R/2})}\right), \tag{5.6}$$

under the assumptions that $R \leqq R_1$, the coefficients of L are smooth, u belongs to $C^\infty(\overline{B}_{3R/2})$, $Lu = f \in C^\infty(\overline{B}_{3R/2})$, and $u \geqq 0$ in $B_{3R/2}$.

We note that if R_1 is chosen small enough, then the maximum principle holds for L in $B_{3R/2}$ (say for $C^2(B_{3R/2})\cap C(\overline{B}_{3R/2})$ functions) and hence there is uniqueness for the Dirichlet problem for L in $B_{3R/2}$. This is a consequence of Corollary 1.7 — see (a) in Remark 2.4 — or of a more basic and well-known result: the maximum principle for "narrow" domains; see Section 3.3 in [6]. It follows, by virtue of the Fredholm alternative and $W^{2,n}$ estimates (see Theorem 9.14 in [6]), that there exists a unique solution $v \in C^\infty(\overline{B}_{3R/2})$ of

$$\begin{cases} Lv &= f \quad \text{in } B_{3R/2} \\ v &= 0 \quad \text{on } \partial B_{3R/2}\,. \end{cases}$$

Moreover, we will prove that

$$\sup_{B_{3R/2}} |v| \leqq CR^{2-n/p_o}\,||f||_{L^{p_0}(B_{3R/2})}\,. \tag{5.7}$$

We then consider $w = u - v$, that satisfies $Lw = 0$ in $B_{3R/2}$ and $w \geqq 0$ in $B_{3R/2}$ (since $w \geqq 0$ on $\partial B_{3R/2}$ and the maximum principle holds for L in $B_{3R/2}$). Applying the Krylov-Safonov Harnack inequality (see Corollary 9.25 in [6]) to w and using (5.7), we easily get (5.6).

Finally we prove (5.7); note that $v = 0$ on $\partial B_{3R/2}$ and satisfies

$$Mv = f - cv\,.$$

Corollary 5.1 applied with $B_{3R/2}$ in place of Ω gives

$$\begin{aligned} \sup_{B_{3R/2}} |v| &\leqq CR^{2-n/p_o}\,||f - cv||_{L^{p_0}(B_{3R/2})} \\ &\leqq CR^{2-n/p_o}\,||f||_{L^{p_0}(B_{3R/2})} + CR^2\tilde{b}\sup_{B_{3R/2}}|v|\,, \end{aligned}$$

where p_0 and C depend only on n, c_0, and C_0 — we take R_1 so small that $bR_1 \leqq 1$. If R_1 is also taken small enough so that $CR_1^2\tilde{b} \leqq 1/2$, we get (5.7). The proof of Theorem 1.11 is now finished.

6. The Parabolic Case

Let L and D be now a parabolic operator and a domain of $\mathbb{R}^{n+1}$ as introduced in Part (b) of Section 1.

L will always act on functions in the Sobolev space $W^{2,1}_{n+1,\mathrm{loc}}(D)$, which is the space of functions that belong to $W^{2,1}_{n+1}(H)$ for any bounded open set H such that $\overline{H} \subset D$. $W^{2,1}_{n+1}(H)$ is defined to be the completion of $C^\infty(\overline{H})$ under the norm

$$||u||_{W^{2,1}_{n+1}(H)} = ||\partial_t u||_{L^{n+1}(H)} + \sum_{i,j=1}^n ||\partial_{ij} u||_{L^{n+1}(H)}$$
$$+\sum_{i=1}^n ||\partial_i u||_{L^{n+1}(H)} + ||u||_{C(\overline{H})} .$$

We now recall the concept of parabolic boundary of a general domain $D \subset \mathbb{R}^{n+1}$, as defined in Section 3.3 of [7]. The parabolic boundary $\partial_p D$ of D is the set of all points $(x_0, t_0) \in \partial D$ (boundary as a subset of $\mathbb{R}^{n+1}$) for which there exist a positive number α and a continuous function $x(t) \in \mathbb{R}^n$ defined for $t \in [t_0, t_0 + \alpha]$, such that

$$x(t_0) = x_0 \quad \text{and} \quad (x(t), t) \in D \ \text{ for } t \in (t_0, t_0 + \alpha] .$$

It is easy to check that for a cylindrical domain $\Omega_T = \Omega \times (0, T)$ we have

$$\partial_p \Omega_T = (\Omega \times \{0\}) \cup (\partial\Omega \times [0, T)) .$$

Remark 6.1. Following [7] let us define, for a point $(x_0, t_0) \in D$, the set dominating (x_0, t_0) in D to be the set of all points $(x_1, t_1) \in D$ for which $t_1 < t_0$ and there exists a continuous function $x(t) \in \mathbb{R}^n$ defined for $t \in [t_1, t_0]$ such that

$$x(t_1) = x_1 , \quad x(t_0) = x_0 , \quad \text{and} \quad (x(t), t) \in D \ \text{ for } t \in [t_1, t_0] .$$

We denote the set dominating (x_0, t_0) in D by $D(x_0, t_0)$. It is easy to see that $D(x_0, t_0)$ is a nonempty domain whose closure contains (x_0, t_0) and that $\partial_p[D(x_0, t_0)] \subset \partial_p D$.

We immediately conclude that, under the hypothesis of Theorem 1.12 and for any $(x_0, t_0) \in D$, (1.17) may be replaced by

$$w(x_0, t_0) \leqq \sup_{\partial_p[D(x_0,t_0)]} w^+ + C\, d^{\,n/(n+1)}\, ||f||_{L^{n+1}(D(x_0,t_0))} .$$

The same is true for (1.19) in Theorem 1.15.

In the proof of Theorem 1.15 we will use both Theorem 1.12 and the following result. It is Lemma 1 and Theorem 2 in Section 4.1 of [7].

THEOREM 6.2. (KRYLOV, SAFONOV) *Let us assume that $D = Q_R$, where Q_R is a parabolic cylinder of radius R in $\mathbb{R}^{n+1}$, and $R \leqq R_0$, for some positive constant R_0. Suppose that $c \in L^\infty(Q_R)$ and $|c| \leqq \tilde{b}$ in Q_R, for a positive constant $\tilde{b}$. Let σ and τ be such that $0 < \tau \leqq \sigma < 1$ and $u \in W^{2,1}_{n+1}(Q_R)$ satisfy*

$$u \geqq 0 \ \text{ in } Q_R , \quad Lu \leqq 0 \ \text{ in } Q_R , \quad \text{and} \quad Lu \leqq -1 \ \text{ in } \Gamma ,$$

where Γ is a measurable set of Q_R such that $|\Gamma| \geqq \sigma |Q_R|$. Then

$$\inf_{Q_{\tau R}} u \geqq \mu R^2 ,$$

where μ is a positive constant depending only on $n, c_0, C_0, bR_0, \tilde{b}R_0^2, \sigma$, and τ.

Theorem 6.2 is stated in [7] for the case $c \leqq 0$. Our statement follows from this, since $(M - c^-)u \leqq Lu$ if $u \geqq 0$.

We now point out that if M has smooth coefficients, Q_ρ is a parabolic cylinder, and $h \in L^{n+1}(Q_\rho)$, then the problem

$$\begin{cases} Mv &= h \quad \text{in } Q_\rho \\ v &= 0 \quad \text{on } \partial_p Q_\rho \end{cases}$$

has a unique solution $v \in W^{2,1}_{n+1}(Q_\rho)$. Note that $W^{2,1}_{n+1}(Q_\rho) \subset C(\overline{Q}_\rho)$, as follows immediately from our definition of $W^{2,1}_{n+1}(Q_\rho)$.

The uniqueness of the problem is a consequence of Theorem 1.12. Besides, this theorem gives an L^∞ bound for v. The problem also has an a priori $W^{2,1}_{n+1}(Q_\rho)$ bound for v in terms of the $L^{n+1}(Q_\rho)$-norm of h, as shown in Theorem 17 of [10].

In order to show existence, we approximate h in $L^{n+1}(Q_\rho)$ by C^∞ functions with compact support in Q_ρ, solve the corresponding problems (that have solutions in $C^\infty(\overline{Q}_\rho)$), and apply the a priori estimates. See also Theorem 9.1 in Chapter IV of [8] for this existence and uniqueness result in the more general case of continuous and bounded coefficients.

We will also need the following lemma, which is already proved in [7].

LEMMA 6.3. *Let D be bounded and $w \in C(\overline{D})$ satisfy $w \leqq -\varepsilon$ on $\partial_p D$, for a positive constant ε. Then, for any $\delta > 0$ small enough and for any component D^δ of $\{(x,t) \in D : d((x,t), \partial D) > \delta\}$, where $d(\cdot, \partial D)$ denotes the distance in $\mathbb{R}^{n+1}$ to the boundary of D, we have that*

$$w \leqq -\varepsilon/2 \quad \text{on } \partial_p D^\delta .$$

Proof: By continuity we have that $w \leqq -\varepsilon$ on the closure of $\partial_p D$. Hence, for δ small enough, $w \leqq -\varepsilon/2$ in the δ-neighborhood of $\partial_p D$ lying in D.

The lemma will be proved if we show that any point $(x_1, t_1) \in \partial_p D^\delta$ is at distance δ of $\partial_p D$. It is clear that $d((x_1, t_1), \partial D) = \delta$ and therefore we can take a point $(x_2, t_2) \in \partial D$ at distance δ of (x_1, t_1).

We know that there is a curve $x(t)$ such that $x(t_1) = x_1$ and $(x(t), t) \in D^\delta$ for $t \in (t_1, t_1 + \alpha]$. Then the shifted curve $(x(t), t) + (x_2, t_2) - (x_1, t_1)$ starts at (x_2, t_2) and lies in D except for the starting point. We conclude that $(x_2, t_2) \in \partial_p D$ and hence $d((x_1, t_1), \partial_p D) = \delta$. The lemma is now proved.

Proof of Theorem 1.15: Note that the parabolic boundary of any component of $\{(x,t) \in D : \tilde{w}(x,t) := w(x,t) - \sup_{\partial_p D} w^+ > 0\}$ is contained in $\{(x,t) \in D : \tilde{w}(x,t) = 0\} \cup \partial_p D$. Therefore we can reduce our problem to the case $c \equiv 0$, as in

the proof of Theorem 1.4, and conclude that it is enough to prove

$$\sup_D w \leqq CR^{n/(n+1)} \, ||f||_{L^{n+1}(D)} \tag{6.1}$$

when

$$Mw \geqq f \text{ in } D\,, \quad \sup_{\partial_p D} w \leqq 0\,, \quad \text{and} \quad 0 < \sup_D w < \infty\,. \tag{6.2}$$

Case 1. We first assume that D is bounded.

Take any $\varepsilon > 0$ and apply Lemma 6.3 to $w - \varepsilon$. We have, for δ sufficiently small,

$$w - \varepsilon \leqq -\varepsilon/2 \text{ on } \partial_p D^\delta \tag{6.3}$$

and $\overline{D^\delta} \subset D$. We see that it suffices to prove (by letting $\varepsilon \to 0$ at the end)

$$\sup_{D^\delta}(w - \varepsilon) \leqq CR^{n/(n+1)} \, ||f||_{L^{n+1}(D^\delta)}\,.$$

By (6.3), we can approximate $w - \varepsilon$ in $W^{2,1}_{n+1}(D^\delta)$ by smooth functions which are nonpositive on $\partial_p D^\delta$. We conclude that it is enough to prove

$$\sup_{D^\delta} w \leqq CR^{n/(n+1)} \, ||f||_{L^{n+1}(D^\delta)} \tag{6.4}$$

when $w \in C^\infty(\overline{D^\delta})$, $Mw \geqq f$ in D^δ, and $\sup_{\partial_p D^\delta} w \leqq 0$.

We now approximate the coefficients of M in $L^{n+1}(D^\delta)$ by smooth ones, which we extend smoothly to $\mathbb{R}^{n+1}$. Since $w \in C^\infty(\overline{D^\delta})$, we can pass to the limit in (6.4), and conclude that it suffices to prove (6.1) when the coefficients of M are smooth in $\mathbb{R}^{n+1}$, (1.14) and (1.15) hold for all $(x,t) \in \mathbb{R}^{n+1}$, and

$$w \in C^\infty(\overline{D})\,, \quad Mw \geqq f \text{ in } D\,, \quad w \leqq 0 \text{ on } \partial_p D\,, \quad \text{and} \quad \sup_D w > 0\,. \tag{6.5}$$

Hence we now assume the smoothness of the coefficients and (6.5); we need to show (6.1). Let us fix $\eta > 0$ and consider

$$K := \sup_{\overline{D}} w > 0\,.$$

We take $(y,s) \in D$ such that $K - \eta \leqq w(y,s)$, and, since R satisfies condition (PG) relative to σ, τ, D, a parabolic cylinder $Q_{R_{(y,s)}} = Q_\rho$ (we set $R_{(y,s)} = \rho$ to simplify notation) such that

$$(y,s) \in Q_{\tau\rho} \quad \text{and} \quad |Q_\rho \setminus D^*| \geqq \sigma |Q_\rho|\,, \tag{6.6}$$

where $D^* := D_{(y,s)}$ is the component of $Q_\rho \cap D$ containing (y,s).

Consider the function

$$u = K - w .$$

We have that $u \geqq 0$ in $\overline{D}$, $u \geqq K$ on $\partial_p D$, and $u(y,s) \leqq \eta$.

As we have pointed out, the problem

$$\begin{cases} Mu_1 &= -(Mw)^- \chi_{D^*} & \text{in } Q_\rho \\ u_1 &= 0 & \text{on } \partial_p Q_\rho , \end{cases}$$

has a unique solution $u_1 \in W^{2,1}_{n+1}(Q_\rho)$. We now apply Theorem 1.12 to get, since $(Mw)^- \leqq |f|$ and $\rho = R_{(y,s)} \leqq R$,

$$0 \leqq u_1 \leqq C_1 R^{n/(n+1)} \, ||f||_{L^{n+1}(D)} \quad \text{in } \overline{Q}_\rho , \tag{6.7}$$

where C_1 depends only on n, c_0, and bR_0.

We can also solve the problem

$$\begin{cases} Mu_2 &= -\chi_{Q_\rho \setminus D^*} & \text{in } Q_\rho \\ u_2 &= 0 & \text{on } \partial_p Q_\rho , \end{cases}$$

with $u_2 \in W^{2,1}_{n+1}(Q_\rho)$. By Theorem 1.12,

$$0 \leqq u_2 \leqq C_2 \rho^2 \quad \text{in } \overline{Q}_\rho ,$$

where C_2 depends only on n, c_0, and bR_0. By (6.6) we can apply Theorem 6.2 to u_2 and M in Q_ρ to get

$$u_2(y,s) \geqq \mu \rho^2 , \tag{6.8}$$

where $\mu > 0$ depends only on $n, c_0, C_0, bR_0, \sigma$, and τ.

It is easy to check that

$$\begin{cases} M(u + u_1) &\leqq 0 & \text{in } D^* \\ u + u_1 &\geqq 0 & \text{on } \partial_p Q_\rho \cap \overline{D} \\ u + u_1 &\geqq K & \text{on } Q_\rho \cap \partial_p D \end{cases}$$

and

$$\begin{cases} Mu_2 &= 0 & \text{in } D^* \\ u_2 &= 0 & \text{on } \partial_p Q_\rho \cap \overline{D} \\ u_2 &\leqq C_2 \rho^2 & \text{on } Q_\rho \cap \partial_p D . \end{cases}$$

Note that u, u_1, and u_2 belong to $W^{2,1}_{n+1}(D^*) \cap C(\overline{D^*})$ and that $\partial_p D^* \subset \partial_p(Q_\rho \cap D) \subset (\partial_p Q_\rho \cap \overline{D}) \cup (Q_\rho \cap \partial_p D)$. Hence we can apply the maximum principle for M in D^* — which is a consequence of Theorem 1.12 — to get

$$u + u_1 \geqq \frac{K}{C_2 \rho^2} u_2 \quad \text{in } D^* .$$

This inequality evaluated at (y,s) gives, with the aid of (6.7) and (6.8),

$$C_1 R^{n/(n+1)} \, ||f||_{L^{n+1}(D)} \geqq u_1(y,s) \geqq (u+u_1)(y,s) - \eta \geqq (\mu/C_2)K - \eta \,,$$

that, letting $\eta \to 0$, implies estimate (6.1).

Case 2. Assume finally that D is unbounded.

Let w satisfy (6.2). Before approximating w and the coefficients of M by smooth functions, we consider $\eta, K, (y,s)$, and Q_ρ as in Case 1. Note that in the rest of the proof (for Case 1), $w \leqq 0$ on $Q_\rho \cap \partial_p D$ — rather than $w \leqq 0$ on $\partial_p D$ — is needed. We now take $\varepsilon > 0$. The proof of Lemma 6.3 (applied to $w - \varepsilon$) shows that δ may be taken small enough such that $w - \varepsilon \leqq -\varepsilon/2$ on $Q_\rho \cap \partial_p D^\delta$. We now approximate $w - \varepsilon$ and the coefficients of M in $Q_\rho \cap D^\delta$ — whose closure is compact and contained in D — by smooth functions. Since $Q_\rho \cap \partial_p(Q_\rho \cap D^\delta) \subset Q_\rho \cap \partial_p D^\delta$, we can take the smooth function w_ε that approximates $w - \varepsilon$ to be nonpositive in $Q_\rho \cap \partial_p(Q_\rho \cap D^\delta)$.

We proceed as in Case 1, now applied to w_ε. We get the desired estimate by letting $\varepsilon \to 0$ first, and then $\eta \to 0$. Theorem 1.15 is now proved.

Remark 6.4. We claim that if $Q_{R_{(x,t)}}$ is a parabolic cylinder as in Definition 1.13 then

$$Q_{R_{(x,t)}} \cap \partial_p D \neq \varnothing \,.$$

This is because $\tau \leqq \sigma$ in Definition 1.13.

Our claim is shown as follows. Set $\rho = R_{(x,t)}$, to simplify notation. Let $Q_\rho = B_\rho \times (s - \rho^2, s)$ and denote $D_{(x,t)}$ by D^*. Since

$$|Q_\rho \setminus D^*| \geqq \sigma |Q_\rho| > \tau^2 |Q_\rho| = |\, B_\rho \times (s - \tau^2\rho^2, s) \,| \,,$$

we have that $Q_\rho \setminus D^* \not\subset B_\rho \times (s - \tau^2\rho^2, s)$.

Hence there is a point $(x_1, t_1) \in Q_\rho \setminus D^*$ such that $t_1 \leqq s - \tau^2\rho^2$.

By (1.18), $(x,t) \in Q_\rho \cap D^*$ and $t > s - \tau^2\rho^2$. Therefore $t_1 < t$. Considering the segment that joints (x,t) and (x_1,t_1), we get that $Q_\rho \cap \partial_p D^* \neq \varnothing$, and hence, $Q_\rho \cap \partial_p D \neq \varnothing$.

Appendix. Proof of Corollary 5.1

We present here the proof given in [4] of Corollary 5.1. The first result that we need is the following.

PROPOSITION A.1. (BAUMAN) *Under the hypothesis of Theorem* 4.1 (*recall that B_R is a ball of radius R such that $B_{4R} \subset \Omega$*), *we have that*

$$\int_{B_R} G(x,y)\,dy \leqq C \int_{B_{R/2}} G(x,y)\,dy \qquad \forall\, x \in \Omega \,,$$

with C as in Theorem 4.1.

The previous proposition is a particular case of the "doubling property" for non-negative adjoint supersolutions, a result also due to Bauman; see [1]. It is also stated and proved in [4] (see Lemma 2.0; their short proof for operators containing only second-order terms is easily extended to our operator M).

It is immediate to see that the Fabes and Stroock reversed Hölder inequality for the Green's function (Theorem 4.1), Proposition A.1 (used repeatedly) and a covering argument imply that the reversed Hölder inequality for the Green's function also holds when the ball B_R is replaced by a cube Q_R. More precisely, assume the hypothesis of Theorem 4.1 with the condition $B_{4R} \subset \Omega$ replaced by $Q_{2R} \subset \Omega$, where Q_R is a cube of $\mathbb{R}^n$ with side-length R and Q_{2R} is the cube with same center as Q_R and double side-length (the symmetric double of Q_R). Then

$$\text{(A.1)} \qquad \left(\frac{1}{R^n}\int_{Q_R} G(x,y)^{n/(n-1)}\,dy\right)^{(n-1)/n} \leqq C\,\frac{1}{R^n}\int_{Q_R} G(x,y)\,dy\,,$$

for any $x \in \Omega$, with C as in Theorem 4.1.

Inequality (A.1) combined with the following result by Gehring — it is Lemma 3 in [5] — will easily imply Corollary 5.1.

THEOREM A.2. (GEHRING) *Assume that $s > 1$ and $C > 1$ are constants, Q is a cube in $\mathbb{R}^n$, and g is a non-negative function which belongs to $L^s(Q)$ and satisfies*

$$\left(\frac{1}{|Q'|}\int_{Q'} g^s\right)^{1/s} \leqq C\,\frac{1}{|Q'|}\int_{Q'} g\,,$$

for each parallel cube $Q' \subset Q$. Then there exists a number q depending only on n, s, and C such that $q > s$, $g \in L^q(Q)$ and

$$\left(\frac{1}{|Q|}\int_{Q} g^q\right)^{1/q} \leqq 2\left(\frac{1}{|Q|}\int_{Q} g^s\right)^{1/s}.$$

We can now prove Corollary 5.1. Take a cube Q of side-length $2\mathrm{diam}(\Omega) \leqq 2d$ and a ball B of radius $2n^{1/2}\mathrm{diam}(\Omega) \leqq 2n^{1/2}d$ such that $\Omega \subset Q \subset 2Q \subset B$ (where $2Q$ denotes the symmetric double of Q). Let $\tilde{G}$ be the Green's function corresponding to M and B. We have that $G \leqq \tilde{G}$ in $\Omega \times \Omega$, by the maximum principle. The ABP estimate implies that, for any $x \in \Omega$, $\tilde{G}(x,\cdot) \in L^{n/(n-1)}(B) \subset L^{n/(n-1)}(Q)$. By (A.1) we know that the reversed Hölder inequality for $\tilde{G}$ holds on any cube $Q' \subset Q$. Hence we can apply Theorem A.2 to $g = \tilde{G}(x,\cdot)$, $s = n/(n-1)$, and Q. We conclude the existence of p_0 and C, as in Corollary 5.1, such that

$$\left(\int_{\Omega} G(x,y)^{q_0}\,dy\right)^{1/q_0} \leqq \left(\int_{Q} \tilde{G}(x,y)^{q_0}\,dy\right)^{1/q_0}$$

$$\leqq C \operatorname{diam}(\Omega)^{1-n/p_0} \left(\int_Q \tilde{G}(x,y)^{n/(n-1)}\, dy \right)^{(n-1)/n}$$
$$\leqq C \operatorname{diam}(\Omega)^{2-n/p_0},$$

where q_0 is the conjugate exponent of p_0.

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