Elliptic PDEs in Probability and Geometry. Symmetry and regularity of solutions

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Abstract

We describe several topics within the theory of linear and nonlinear second order elliptic Partial Differential Equations. Through elementary approaches, we first explain how elliptic and parabolic PDEs are related to central issues in Probability and Geometry. This leads to several concrete equations. We classify them and describe their regularity theories. After this, most of the paper focuses on the ABP technique and its applications to the classical isoperimetric problem —for which we present a new original proof—, the symmetry result of Gidas-Ni-Nirenberg, and the regularity theory for fully nonlinear equations.

Contents

1 Introduction

The goal of this paper is to introduce the reader —expert or not— to some important techniques and results in the theory of second order elliptic PDEs. Its main features and objectives are the three following.

- To explain through elementary approaches how elliptic PDEs arise in Probability and Geometry. For instance, we show how simple is the relation between probabilistic issues on random walks and the Laplace operator —and also other elliptic operators, as well as the heat equation.
- To describe, giving full proofs of most results, a useful technique called the ABP method, as well as its applications to: a new original proof of the classical isoperimetric problem, the symmetry result of Gidas-Ni-Nirenberg, and the $C^{2,\alpha}$ regularity theory for fully nonlinear elliptic equations.
- To provide a larger picture of other techniques and results for elliptic PDEs in an informative way. We focus mainly on regularity theories for other types of equations (divergence form or variational equations), viscosity solutions, controlled diffusion processes, and PDEs on Riemannian manifolds.

The table of contents gives a fast account of the topics that we treat.

In section 2, several easy to state, but central problems in Probability and Geometry have been selected. Through elementary approaches, we show how they lead to several PDEs. With very simple arguments —sometimes heuristic but always containing essential facts—, we show how simple is the relation between random walks in Probability and the Laplace operator. More general elliptic operators, as well as the heat equation, are similarly obtained.

These motivating problems and arguments lead to various PDEs, which we classify in section 3. This section also explains the basic strategy towards regularity for nonlinear elliptic equations and the need for two different linear theories: the "divergence form" or variational theory, and the "nondivergence form" theory. These two theories usually serve to get some crucial regularity for the solutions of quasilinear and fully nonlinear equations, respectively. Once this is achieved, further regularity may be obtained using the classical Schauder and Calderón-Zygmund theories —briefly described in subsection 5.4. The topics treated in the rest of the paper concern mainly the nondivergence theory.

Section 4 describes in detail a nondivergence technique called the ABP method. It was introduced by Alexandroff in the sixties to study the curvature of manifolds and the solutions of elliptic equations. We use the ABP method to give a new and simple proof of the classical isoperimetric problem in $\mathbb{R}^n$. We also use it to establish the maximum principles developed by Berestycki and Nirenberg near 1991 to improve the moving planes method. Using their improved version, we present the full proof of the symmetry result of Gidas-Ni-Nirenberg.

The ABP estimate also plays a crucial role in the Krylov and Safonov theory from 1979, which established the Harnack inequality for linear uniformly elliptic equations with measurable coefficients, written in nondivergence form $a_{ij}(x)\partial_{ij}u = f(x)$. This important result allowed the development of a regularity theory for fully nonlinear equations. Evans and Krylov obtained independently, near 1982, a $C^{2,\alpha}$ interior estimate for convex equations $F(D^2u) = 0$. We present Caffarelli's simple proof of this estimate in subsection 5.2.

The ABP estimate, the Krylov-Safonov theory, and the regularity theory for fully nonlinear elliptic PDEs are described —with almost fully detailed proofs— in subsections 5.1 and 5.2. The rest of section 5 has a rather informative tone. It is a fast introduction to the notions of fully nonlinear elliptic operator, examples from controlled diffusion processes (this is related with the probabilistic issues described in the beginning of the paper), the theory of viscosity solutions, Caffarelli's extension of the linear Schauder and Calderón-Zygmund theories to the fully nonlinear context, and finally the study of elliptic PDEs (in particular the Laplace-Beltrami operator) and optimal transport maps on Riemannian manifolds.

Further reading. Evans [26] and Salsa [49] contain excellent expositions of the relations between Probability and elliptic and parabolic equations. They use simple and informative approaches. A beautiful article by Grigor'yan [34] explains in detail the relations beetwen Brownian motion, the Laplace operator, and the heat equation, all of it within the general framework of Riemannian manifolds.

The book by Bass [2] is concerned with probabilistic techniques for elliptic PDEs. It includes the original proofs of the Krylov-Safonov theory, which used probabilistic tools.

Regarding the purely analytical approach to the ABP estimate, the Krylov-Safonov Harnack inequality, and the regularity theory for fully nonlinear elliptic equations, the lecture notes of Evans [24] are a very good introduction which complements the present paper. Another complementary reference on these topics —and also on further symmetry results— is the author's paper [9]. For further results and more detailed proofs in regularity issues we will refer to the books by Caffarelli-Cabré [14], Gilbarg-Trudinger [32], and Han-Lin [36].

Symmetry questions for elliptic PDEs are treated nicely in the papers by Berestycki-Nirenberg [3] and by Brezis [5].

For the non expert reader in elliptic PDEs, the book [25] by Evans will be very useful. The book by Chavel [19] is a great introduction to Riemannian manifolds and isoperimetric inequalities.

2 PDEs in Probability and Geometry: elementary approach

2.1 Probability of hitting the exit. Expected hitting time

2.1.1 Probability of hitting the exit and harmonic functions

Let $\Omega \subset \mathbb{R}^2$ be a domain and suppose that its boundary is decomposed into two parts: Γ_o (the opened or exiting part of the boundary) and Γ_c (the closed part). We have $\partial\Omega = \Gamma_o \cup \Gamma_c$ and $\Gamma_o \cap \Gamma_c = \emptyset$. A particle is located at a point $(x, y) \in \Omega$ and starts moving or "walking" in a random way. The "floor" Ω is flat and homogeneous, and the particle does not privilege any special direction. In addition, at every time the particle moves independently of its past history.

We want to compute the probability $u(x, y)$ of hitting the opened part Γ_o (and hence exiting the domain) the first time that the particle hits the boundary $\partial\Omega$.

A clever and simple way to solve the problem is the following. We approximate the movement by random increments of step h, with $h > 0$ small. From (x, y) the particle can move to $(x+h, y)$, $(x-h, y)$, $(x, y+h)$, or $(x, y-h)$, each one with probability $1/4$. Starting at (x, y), let $u^h(x, y)$ be the corresponding probability of hitting the exit Γ_o (the first time that $\partial\Omega$ is hitted) when the particle moves on the lattice of side h. It is natural to expect that $u = \lim u^h$ as $h \to 0$.

The formula of conditional probabilities gives immediately that

$$u^h(x, y) = \frac{1}{4}\left\{u^h(x+h, y) + u^h(x-h, y) + u^h(x, y+h) + u^h(x, y-h)\right\}, \tag{2.1}$$

and hence

$$\begin{aligned} &\left\{u^h(x+h, y) + u^h(x-h, y) - 2u^h(x, y)\right\} + \\ &\quad + \left\{u^h(x, y+h) + u^h(x, y-h) - 2u^h(x, y)\right\} = 0. \end{aligned} \tag{2.2}$$

Dividing by h^2 and assuming a good enough convergence of u^h towards u, we obtain

$$\Delta u(x, y) := u_{xx}(x, y) + u_{yy}(x, y) = 0. \tag{2.3}$$

That is, $u : \Omega \longrightarrow \mathbb{R}$ is a *harmonic function*! In addition, we also expect its boundary values to be given by the following problem.

$$u(p) = \text{probability of hitting } \Gamma_o \text{ starting at } p$$

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \begin{cases} 1 & \text{on } \Gamma_o \\ 0 & \text{on } \Gamma_c. \end{cases} \end{cases} \tag{2.4}$$

This Dirichlet boundary value problem can be shown to have a unique solution (in an appropriate sense) with values in $(0, 1)$. Its solution can be computed numerically and, in many cases, even explicitly with the aid of conformal mappings. In subsections 2.1.2 and 2.2 we make comments on some rigorous justifications of the convergence as $h \to 0$ of the discrete solutions corresponding to walks on the lattices towards u.

Note that in $\mathbb{R}^n$, the discretization method described above leads in the same simple way to the Laplace operator in higher dimensions and to exiting probabilities being harmonic functions.

Another way to understand this strong relation between probabilities and the Laplacian is through the *mean value property*. In the problem above, assume that a closed ball $\overline{B}_r(p)$ of radius r and centered at a point p is contained in Ω. Starting at p, the probability density of hitting first a given point on the sphere $\partial B_r(p)$ is constant on the sphere —that is, it is uniformly distributed on the sphere. Therefore, the probability $u(p)$ of exiting through Γ_o starting at p is the average of the exiting probabilities u on the sphere, by the formula of conditional probabilities. That is, u satisfies the mean value property on spheres:

$$u(p) = \frac{1}{|\partial B_r(p)|} \int_{\partial B_r(p)} u$$

for every $p \in \Omega$, with r small enough. Now, it is well known that this leads to u being harmonic.

It is remarkable that the probability u in our problem coincides with the temperature or heat that we feel at $p = (x, y)$ when we keep an infinite time the boundary heated at 1 degree on Γ_o and 0 degrees at Γ_c —and there is no heat source on the floor Ω. See next subsection for more comments on the heat equation.

The book by Salsa [49] contains more details and further facts in the spirit of this subsection.

2.1.2 Expected hitting time, costs, and the Poisson problem

A second motivating problem is the following. In the previous situation, now we would like to compute a certain expected "time" —that we also denote by $u(x, y)$— spent by the particle starting at (x, y) before hitting the boundary $\partial\Omega$ for the first time. Here we look at the whole $\partial\Omega$, which is no longer divided into two parts as before. We can proceed computing the time of the random walk at the discrete level, as above. This amounts to adding a constant, which depends on the step h, to the right hand side of (2.1). That is, we have

$$\begin{aligned} u^h(x, y) &= T(h) + \\ &+ \frac{1}{4}\left\{u^h(x+h, y) + u^h(x-h, y) + u^h(x, y+h) + u^h(x, y-h)\right\}. \end{aligned} \tag{2.5}$$

The constant $T(h)$ is the time spent by the particle on one single step, horizontal or vertical, of length h. Proceeding as above, and since we need to divide (2.5) by h^2, we see that $T(h)$ must be of order h^2. Choosing

$$T(h) = h^2/2$$

and letting $h \to 0$, we are led to

$$u(p) = \text{expected time to hit } \partial\Omega \text{ starting at } p$$

$$\begin{cases} -\frac{1}{2}\Delta u &= 1 \quad \text{in } \Omega \\ \quad u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Its solution $u(x, y)$ is the expected time to hit $\partial\Omega$ starting at (x, y). That the time u on the boundary $\partial\Omega$ equals 0 is reasonable but only heuristic at this level. We have also discovered that increments of time are to be taken proportional to the square of space increments —exactly as in the scaling for the heat equation in subsection 2.2 below.

A rigorous justification of the convergence of u^h towards the solution u of (2.6) can be given as in the numerical theory of *finite difference schemes* for PDEs. One considers the discrete Laplacian Δ_h, defined by the expression in the left hand side of (2.2) divided by h^2. From (2.5) with $T(h) = h^2/2$, we have that

$$-\frac{1}{2}\Delta_h u^h = 1.$$

Now one computes $\Delta_h(u^h - u)$, approximating $\Delta_h u$ by Δu using Taylor's formula and controlling the error (of order h) using elliptic estimates for the derivatives of the exact solution u. Finally, one applies a maximum estimate for $u^h - u$ from the knowledge of $\Delta_h(u^h - u)$ —see for instance chapter 9 of [37] for details.

The limit as $h \to 0$ of the paths described by the random movement on the lattices leads to the central subject of *Brownian Motion*. It originated in 1827, when the botanist Robert Brown observed this type of random movement in pollen particles floating in water.

From the previous probabilistic interpretations for the solutions of problems (2.4) and (2.6), one can conceive that the solution of the general Poisson problem

$$\begin{cases} -\frac{1}{2}\Delta u &= f(x) \quad \text{in } \Omega \\ \quad u &= g(x) \quad \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

can also be interpreted in probabilistic terms. Indeed, one can write a formula for the solution u in terms of expected values of f and g among all possible Brownian paths. This is the *Feynman-Kac formula* (see chapter 6 of [26] and also [49]). The functions f and g should be thought as costs that one pays, respectively, along the random movement and at the stopping time on the boundary. In this way we see that elliptic PDEs are strongly connected with stochastic control theory (explained below in subsection 5.3.2) and also with Mathematical Finance.

To finish this subsection, let us mention an easy but instructive problem. One can compute explicitly the solution u of (2.4) when

$$\Omega = B_R \setminus \overline{B}_\delta \text{ is an annulus, } \Gamma_o = \partial B_\delta, \text{ and } \Gamma_c = \partial B_R.$$

The expressions differ in case that the dimension $n = 2$ or $n \geq 3$. The limit function as $R \to \infty$ represents the probability of eventually (sooner or later) hitting the ball B_δ when starting from a point outside this ball. The different results that one obtains for $n = 2$ and $n \geq 3$ reflect an important difference between Brownian motion in the plane (being *recurrent*) and in higher dimensions (being *transient*). There is a corresponding analytical result to this fact. It is related with the different properties that subharmonic functions have in dimension 2 and in higher dimensions —see [34, 49].

2.1.3 Anisotropic media. General elliptic operators

Suppose now that the medium Ω is neither isotropic (that is, it is directionally dependent) nor homogeneous (that is, it differs from point to point). We can conceive a random discrete movement as follows. We move from (x, y) to four possible points at distance h located at two orthogonal axis forming a given angle α with the horizontal, and with different probabilities $q/2$ and $(1-q)/2$ for the two points on one axis and the two points on the other axis. The angle α and the probability $q/2$ depend on (x, y). After the same analysis as above, we encounter now the elliptic equation

$$Lu := a_{ij}(x, y)\partial_{ij} u = 0. \tag{2.8}$$

Using a standard convention, we do not write the summations over repeated indexes, such as $\Sigma_{i,j=1}^2$ in (2.8). The positive definite symmetric matrix of coefficients $a_{ij}(x, y)$ can be computed explicitly from the angle α and probability data $q/2$ given at the point (x, y).

The linear operator L is called the infinitesimal generator associated to a *diffusion or Markov process* describing the random movement. See [2, 51] for expositions.

As mentioned before, the same problem may be posed in $\mathbb{R}^n$. The discretization method above leads in the same way to the elliptic operators (2.8) posed in $\mathbb{R}^n$.

2.2 The heat equation, Gaussians, and the Central Limit Theorem

Consider now that a particle moves in a two dimensional lattice as follows. The particle being at (x_0, t_0), each increment of time $\delta t := h^2$ it moves an increment of space $\delta x = h$ and goes to $(x_0 - h, t_0 + h^2)$ or to $(x_0 + h, t_0 + h^2)$, each one with probability $1/2$. We will see that the choice $\delta t := h^2$ is made to ensure that a certain limit as $h \to 0$ exists.

The particle starting at $(0, 0)$, let $u^h(x, t)$ be the probability that the particle is at x at time t, where $t > 0$ and (x, t) is a point in the space-time h-lattice.

As in the previous subsection on hitting probabilities, conditional probabilities give

$$u^h(x,t) = \frac{1}{2}\left\{u^h(x-h,t-h^2) + u^h(x+h,t-h^2)\right\}.$$

From this relation we subtract $u^h(x,t-h^2)$ in both sides and divide by h^2. Letting $h \to 0$ and assuming good enough convergence to a function u, we are lead to the heat equation

$$\left(u_t - \frac{1}{2}\Delta u\right)(x,t) = 0, \qquad x \in \mathbb{R}, t > 0,$$

where $u(x,t)$ is now the probability density that a particle is at x at time $t > 0$ having started at $x = 0$ at time $t = 0$. In particular, our initial condition at $t = 0$ for this PDE is the delta of Dirac at the starting point $x = 0$. The solution of such initial value problem for the heat equation is called the fundamental solution. It is well known to be given by the *Gaussian*

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}. \tag{2.9}$$

Note that the function u^h can also be thought as describing the *propagation of random errors* $\pm h$. The total error (or the position of the particle in the previous particle interpretation) at time $t = Nh^2$ can be written as

$$x(t) = x(Nh^2) = hX_1 + \cdots + hX_N, \tag{2.10}$$

where $X_1, \cdots, X_N$ are independent random variables, all of them equally distributed with values ± 1 each one with probability $1/2$. Taking $t = 1$ we have that $h = 1/\sqrt{N}$ in (2.10). Since we heuristically know that the limiting function u is the Gaussian (2.9) with $t = 1$, we deduce

$$\frac{X_1 + \cdots + X_N}{\sqrt{N}} \longrightarrow \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

This is simply the statement of the *Central Limit Theorem* —for the case of our particular random variables.

Evans [26] and Salsa [49] are nice introductory expositions to these topics. [26] contains a more careful analysis of passing to the limit as $h \to 0$ using the Laplace-De Moivre Theorem.

2.3 Images and pixels

Suppose now that $u(x,y)$ is the level of gray at a point (x,y) in a black and white picture or image. We have that $u \in [0,1]$ and that the value 0 corresponds to black and the value 1 to white. The discretization with the squared lattice of side h corresponds now to the pixels in a digital image. The level of gray is now given by a function u^h. If one pixel is lost or unknown, a natural way to assign to it a gray level is making the average of the gray levels of the four adjacent pixels. This leads to formula (2.1) for u^h, and hence to the limit function u being *harmonic*.

An interesting question is the following. Assume now that the gray levels of several pixels are unknown. For instance, three pixels could be unknown: a certain pixel, the one on its left, and the one below it. Assume that all adjacent pixels to these three are known. Using the previous method of averaging the four adjacent pixels to resolve an unknown pixel, is the problem well posed —so that the three unknown pixels can be found uniquely?

The answer is affirmative. One finds a system of three linear equations for the three values of u^h at the unknown pixels. The corresponding matrix (this is the matrix associated to the discrete Laplacian Δ_h) turns out be symmetric and positive definite —hence invertible. The same matrices appear of course in the numerical method of finite differences.

See [17] for more involved interpolating algorithms for the unknown pixels. They may take into account the level curves of gray when averaging. They lead to other elliptic equations, in these cases degenerate elliptic, such as

$$D^2u(Du^\perp, Du^\perp) = 0,$$
$$D^2u(Du, Du) = 0.$$

2.4 Soap bubbles: minimal surfaces

We turn now to the topic of soap bubbles or minimal surfaces. That is, a surface for which its area within a compact region increases when the surface is perturbed or modified within that region.

Locally, every surface or variety can be written as the graph of a function $y = u(x)$, where $u : H \subset \mathbb{R}^n \to \mathbb{R}$ ($n = 2$ for the case of surfaces). The area functional is given by

$$I(u) = \int_H \sqrt{1 + |Du|^2}\, dx. \tag{2.11}$$

We allow perturbations $u + \varepsilon\varphi$ of u within H, and hence $\varphi \equiv 0$ on ∂H. By minimality, the first variation $(d/d\varepsilon)I(u + \varepsilon\varphi)$ at $\varepsilon = 0$ must be zero —and the second variation nonnegative. It is simple to compute the first variation (or Euler-Lagrange equation of the functional I) obtaining

$$\partial_i \left\{ \frac{\partial_i u}{\sqrt{1 + |Du|^2}} \right\} = 0 \qquad \text{in } H \subset \mathbb{R}^n. \tag{2.12}$$

This is a nonlinear elliptic equation —the equation for *minimal graphs*. The left hand side of (2.12) is the mean curvature of the graph of u. It must be identically zero for minimal varieties. The book of Giusti [33] is a great monograph on minimal surfaces.

2.5 The isoperimetric problem

The solution of the isoperimetric problem states that, among all regular enough bounded regions in $\mathbb{R}^n$ with same perimeter (or $n-1$ dimensional measure) the balls, and only the balls, have the maximum volume (or n dimensional measure).

In section 4 we will give a new and simple proof of this fact using elliptic PDEs. Let us give here several "hints" which indicate that balls should be the solution of the isoperimetric problem.

Suppose that $\Omega \subset \mathbb{R}^n$ is a smooth solution of the isoperimetric problem. Write any small enough portion of its boundary $\partial\Omega$ as the graph of a function $u : H \subset \mathbb{R}^{n-1} \to \mathbb{R}$. As in the previous subsection we consider perturbations of this part of the boundary of Ω. They correspond to the graphs of $u + \varepsilon\varphi$ with $\varphi \equiv 0$ on ∂H, but now with the restriction that the volume of the perturbed domains are all the same. Since the perturbed domain corresponds to the part under the graph, we must have $\int_H (u + \varepsilon\varphi)$ to be constant in ε. Therefore, we have the constrain $\int_H \varphi = 0$. In other words, we consider a family of perturbed functions u with the constrain

$$\int_H u = \text{constant}, \tag{2.13}$$

under which we want to minimize the perimeter of the perturbed domains. That is, we want to minimize the area functional (2.11) subject to the constrain (2.13). Making the first variation, the rule of Lagrange multipliers tells us that the variations of (2.11) and of (2.13) must be equal up to a multiplicative constant c. Hence we get the elliptic equation

$$\partial_i \left\{ \frac{\partial_i u}{\sqrt{1 + |Du|^2}} \right\} = c \qquad \text{in } H \subset \mathbb{R}^{n-1}, \tag{2.14}$$

expressing that the mean curvature of $\partial\Omega$ must be constant for a solution Ω of the isoperimetric problem. Of course spheres have constant mean curvature.

On these lines, Alexandroff proved an important theorem about hypersurfaces with constant mean curvature being spheres —see subsection 4.3. For this he introduced the important *moving planes method*. It is a very flexible method that establishes symmetry properties for solutions of elliptic equations. We describe it in detail in subsection 4.3.

The standard proof of the isoperimetric problem (see [4]) uses *Steiner symmetrization*, as follows. It gives a simple good explanation of why balls, and only balls, solve the isoperimetric problem. Suppose (and this is a restriction) that $\Omega \subset \mathbb{R}^n$ is given by

$$\Omega = \{(x', x_n) \in H \times \mathbb{R} : u_1(x') < x_n < u_2(x')\},$$

where $u_1 < u_2$ are two functions in $H \subset \mathbb{R}^{n-1}$. Consider the Steiner symmetrized domain

$$\Omega^* = \left\{ (x', x_n) \in H \times \mathbb{R} : -\frac{u_2 - u_1}{2}(x') < x_n < \frac{u_2 - u_1}{2}(x') \right\}.$$

Note that Ω and Ω^* have the same volume, by Fubinni. Instead, their perimeters are respectively $I(u_1) + I(u_2)$ and $2I((-u_1 + u_2)/2)$, where I is the area functional (2.11) —with dx replaced by dx'. Since this is a convex functional (its integrand $p \mapsto \sqrt{1 + |p|^2}$ is a convex function), we have

$$2I\left(\frac{-u_1 + u_2}{2}\right) \leq I(u_1) + I(u_2),$$

and hence the perimeter of Ω^* is smaller or equal than that of Ω. The standard proof of the isoperimetric problem uses this fact and that if a domain is symmetric with respect to all hyperplanes passing through a point, then it must be a ball.

Isoperimetric inequalities can also be studied for domains on manifolds. They are powerful analytical tools. For instance, every isoperimetric inequality leads, through the use of the coarea formula, to sharp *Sobolev inequalities* —also in the generality of Riemannian manifolds. See the nice books of Chavel [19, 20] for more on these topics.

2.6 Curvature of manifolds

Most problems on the analysis of the curvature of Riemannian manifolds rely on the study of certain nonlinear elliptic equations. We have already encountered the mean curvature operator in (2.12) and (2.14) when studying minimal surfaces and the isoperimetric problem.

Consider now the problem of establishing conditions for a given function $K \in C^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^2$, which guarantee that K is the *Gauss curvature* for some metric g in Ω. We can attack this problem considering metrics g conformally equivalent to the Euclidean standard metric g_0 in $\mathbb{R}^2$, and hence writing $g = e^{2u} g_0$ where $u : \Omega \to \mathbb{R}$. An easy computation shows that K is the Gauss curvature for some conformal metric g to the standard metric g_0 if and only if there exists a solution $u = u(x)$ of

$$\Delta u + K(x) e^{2u} = 0, \qquad x \in \Omega \subset \mathbb{R}^2. \tag{2.15}$$

The same problem can be set in higher dimensions for the *scalar curvature*, obtaining the equation

$$\Delta u + K(x) u^{\frac{n+2}{n-2}} = 0, \qquad x \in \Omega \subset \mathbb{R}^n, \tag{2.16}$$

involving a pure power nonlinearity with the critical Sobolev exponent. Finding solutions of such equation is the famous *Yamabe problem* —see [38, 18] for more details.

On the other hand, the graph of a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ has Gauss curvature $K(x)$ if u is a solution of the equation

$$F(D^2 u, Du, x) := \det D^2 u - K(x)(1 + |Du|^2)^{(n+2)/2} = 0 \qquad \text{in } \Omega. \tag{2.17}$$

This is the *prescribed Gauss curvature equation* for graphs.

2.7 Optimal transport maps

In 1781 Monge proposed the following variational problem. Which is the best way to move a pile of soil or rubble (*déblais* in French) to an excavation or fill (*remblais*), in order to minimize a certain cost or work. In more precise words, let $f_1, f_2 \geq 0$ be two integrable functions with supports $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^n$ respectively, and with the same total mass $\int_{\Omega_1} f_1 = \int_{\Omega_2} f_2$. Let $c : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be a cost function. The problem consists on finding a map $t : \Omega_1 \to \Omega_2$ minimizing the total cost

$$\mathcal{C}(s) := \int_{\Omega_1} c(x, s(x)) f_1(x)\, dx$$

within the class $S(f_1, f_2)$ of measurable maps $s : \Omega_1 \to \Omega_2$ that preserve the measures $f_1(x)dx$ and $f_2(y)dy$, that is, maps such that

$$\int_{\Omega_1} h(s(x)) f_1(x)\, dx = \int_{\Omega_2} h(y) f_2(y)\, dy \tag{2.18}$$

for all continuous functions h with support in Ω_2.

The functions $c(x, y) = |x - y|^p$, where $p > 0$, are typical examples of costs. Monge chose as a cost function the Euclidean distance, $p = 1$. Even in this particular case $p = 1$, two centuries passed until Sudakov proved, in 1979, the existence of a map $t \in S(f_1, f_2)$ minimizing $\mathcal{C}$ —see also the comments in [1]. In general, this map is not unique when $p = 1$.

In 1987 Brenier proved that, for the quadratic cost $c(x, y) = |x - y|^2$, there is a map $t \in S(f_1, f_2)$ that minimizes $\mathcal{C}$. Besides, t is the gradient of a convex function u. Equality (2.18), rewritten for $s = t = \nabla u$, gives

$$\int_{\Omega_1} h(\nabla u(x)) f_1(x)\, dx = \int_{\Omega_2} h(y) f_2(y)\, dy \qquad \text{for all } h \in C(\Omega_2). \tag{2.19}$$

Then we say that $u : \Omega_1 \to \mathbb{R}$ is a solution in the Alexandroff sense of the *Monge-Ampère type equation*

$$f_2(\nabla u(x)) \det D^2 u(x) = f_1(x) \qquad \text{in } \Omega_1, \tag{2.20}$$

since (2.19) and (2.20) are equivalent when ∇u is bijective and C^1 (by the area or change of variables formula).

See [1, 27, 52] for more on optimal transport maps.

3 Types of elliptic PDEs and their regularity theories

3.1 Semilinear, quasilinear, and fully nonlinear equations

There are three important types of second order nonlinear elliptic equations. They are characterized by their successive degree of nonlinearity.

- *Semilinear equations*:
$$\Delta u + f(x,u) = 0, \tag{3.1}$$
such as (2.3), (2.6), and the Gelfand and Yamabe type equations (2.15) and (2.16). Semilinear equations are linear in the first and second derivatives of u —but nonlinear in the value of u due to the reaction term $f(x,u)$.

- *Quasilinear equations*:
$$\partial_i(H^i(Du)) + f(x,u) = 0, \tag{3.2}$$
a model for quasilinear equations (there are more general expressions). We have encountered equations (2.12) and (2.14) involving the mean curvature operator as examples. The operator $\partial_i(H^i(Du)) = H^i_{p_j}(Du)\,\partial_{ij}u$ in (3.2) is not linear in u, but it is linear as a function of the second derivatives of u. Here, $H^i_{p_j}$ denotes the partial derivative of the i-th component of H with respect its j-th variable.

- *Fully nonlinear equations*:
$$F(D^2u, Du, u, x) = 0$$
such as equations (2.17) and (2.20), or the simpler *Monge-Ampère equation*
$$\det D^2u = f(x) > 0. \tag{3.3}$$
Fully nonlinear equations are nonlinear in the second derivatives of u.

Semilinear and quasilinear equations are usually the Euler-Lagrange equation (a notion already encountered in subsections 2.4 and 2.5) for functionals of the form
$$I(u) = \int_\Omega L(Du(x), u(x), x)\,dx,$$
where L is called the *Lagrangian* (see [25, 32] for more details). For example, the semilinear equation (3.1) is the Euler-Lagrange equation of I corresponding to the Lagrangian $L = (1/2)|Du|^2 - F(x,u)$, with F satisfying $F_u = f$. On the other hand, the Lagrangian
$$L = G(Du) - F(x,u) \tag{3.4}$$
leads to the quasilinear equation (3.2) with $H^i = G_{p_i}$. When G is quadratic, we obtain the Laplacian operator. In next subsection we will explain the notion of ellipticity for equations (3.1), (3.2), and (3.3).

3.2 Towards regularity: linearizing nonlinear equations

The basic strategy to prove the regularity of solutions of nonlinear elliptic equations is based on looking at the nonlinear equation for u as a *linear* equation (also for u, or for the derivatives $\partial_k u$ of u) with variable coefficients which depend on the function u itself.

Let us illustrate these ideas with the quasilinear and fully nonlinear examples. If we differentiate equation (3.2), with $H^i = G_{p_i}$, with respect to the variable x_k, we obtain

$$\partial_i \left\{ G_{p_i p_j}(Du(x))\, \partial_j u_k \right\} = \tilde{f}(x) \qquad \text{where } u_k := \partial_k u, \tag{3.5}$$

$\tilde{f}(x) = -\partial_k \{f(x, u(x))\}$. This is a linear equation for $v := u_k$ of the type

$$\operatorname{div}(A(x)\nabla v) = \partial_i \{a_{ij}(x)\, \partial_j v\} = \tilde{f}(x), \tag{3.6}$$

written in divergence form and with coefficients $a_{ij}(x) := G_{p_i p_j}(Du(x))$ which depend on the first derivatives of u. The equation is said to be elliptic if $A(x) = [a_{ij}(x)]$ is a positive definite matrix for every x. This will hold, independently of the regularity of u, if the function G in the Lagrangian (3.4) is strictly convex.

For the Monge-Ampère operator (3.3), differentiating the equation with respect to x_k, we obtain

$$(D^2u(x))^{ij}\, \partial_{ij} u_k = \frac{\partial_k f(x)}{\det D^2u(x)} \qquad \text{where } u_k := \partial_k u$$

and $(D^2u)^{ij}$ denotes the ij-th element of the inverse matrix of D^2u (if such inverse exists). We have used that $(\det D^2u)\,(D^2u)^{ij}$ is the cofactor of $\partial_{ij}u$ in the matrix D^2u, by Cramer's rule. Hence we have obtained a linear equation for u_k. The difference with the quasilinear case is that now it is not written in divergence form, but in the form

$$\operatorname{tr}(A(x)D^2v) = a_{ij}(x)\, \partial_{ij} v = \tilde{f}(x), \tag{3.7}$$

where $v := u_k = \partial_k u$, $A(x) = [a_{ij}(x)] := [(D^2u(x))^{ij}]$, and tr denotes the trace. The equation is elliptic if, for all x, the symmetric matrix $A(x) = (D^2u(x))^{-1}$, or equivalently $(D^2u(x))$, is positive definite. This will be true if u is a strictly convex function. This is the reason why, in elliptic theory, one considers the Monge-Ampère equation (3.3) with $f > 0$ and looks for strictly convex solutions u. Very little is known about equation (3.3) when f changes sign —a fully nonlinear hyperbolic equation.

Note that we have obtained linear equations with variable coefficients, (3.6) and (3.7), for the first derivatives of the solution of a nonlinear equation. Now, the key point is that we cannot assume the coefficients a_{ij} to be regular, since they depend on the first or second derivatives of u —and precisely the regularity of such derivatives is what we are trying to prove. This is the reason why it is essential to develop regularity theories for linear equations with "measurable" coefficients. We write "measurable" within quotes because what one really does is to assume the coefficients to be regular enough and then, under this hypothesis, prove estimates (for the solution u or $\partial_k u$ of the linear equation) independent of the modulus of regularity assumed a priori for the coefficients.

Summarizing, we have seen the interest of considering linear equations with measurable coefficients, and that the underlying linear theories (which we start describing in next subsection) deal with different objects, (3.6) and (3.7). Hence, we consider linear equations posed in a domain $\Omega \subset \mathbb{R}^n$, of the form

$$\mathcal{L}u := \partial_i(a_{ij}(x)\partial_j u) = f(x) \qquad \text{(divergence form)}, \tag{3.8}$$

that already appeared in (3.6) when differentiating quasilinear equations, and of the form

$$Lu := a_{ij}(x)\partial_{ij}u = f(x) \qquad \text{(nondivergence form)}, \tag{3.9}$$

that appeared in (3.7) when differentiating the Monge-Ampère equation and also in (2.8) when studying random walks in inhomogeneous media. We always assume that $\mathcal{L}$ and L are *uniformly elliptic*, i.e., that

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \qquad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n, \tag{3.10}$$

for some constants $0 < \lambda \le \Lambda$.

Note that (3.8) can be rewritten as $a_{ij}(x)\,\partial_{ij}u + (\partial_i a_{ij}(x))\,\partial_j u = f(x)$, which is an equation of the type (3.9) with first order terms, but that this does not make sense if a_{ij} are only measurable —or if we want estimates independent of the modulus of continuity or differentiability of the coefficients a_{ij}.

3.3 Overview of the regularity theories

Semilinear and quasilinear equations. As we have seen, integration by parts when writing the Euler-Lagrange equation for the functional I leads to a divergence structure of the operator. These equations are hence called *elliptic equations in divergence form*, or also *variational elliptic equations*. Already in the sixties they had a uniqueness, existence, and regularity theory. The existence results are based on the classical methods of the Calculus of Variations, that is, critical point theory for the functional I —usually called energy functional in PDE terminology. The notion of weak solution (distributional solutions and energy solutions) is based on the integration by parts method. The questions about the regularity of solutions have in the Sobolev inequality a powerful tool, and they have been developed by many authors during the twentieth century. See [32, 50, 18] and references therein.

For semilinear equations, where the second order term has constant or regular coefficients, regularity of the solution u is obtained right away from the Schauder and Calderón-Zygmund theories (described below in subsection 5.4) once u, and then the right hand side $f(x,u)$, is known to be bounded. The boundedness of u can be proved for some reaction nonlinearities (those with subcritical or critical growth) through the use of a *bootstrap* technique (see an appendix of [50]). More delicate bounds can be obtained through the *blow-up technique* combined with *Liouville theorems* for solutions in all space, as in the important works of Caffarelli, Gidas, and Spruck [30, 31, 15].

For most quasilinear equations, a stronger linear theory —for divergence form equations (3.8) with measurable coefficients— is needed. The crucial result here was developed in the sixties by De Giorgi, Nash, and Moser (see [32, 36]). They established the Hölder regularity (and a Harnack inequality) for the solutions u of (3.8) under the only assumption of uniform ellipticity for the coefficients.

However, for many quasilinear equations it is a nontrivial issue to ensure the uniform ellipticity of the linearizations of the equation. This is the case for instance of the p-Laplacian operator (see Appendix E in [48]). It is also the case for the mean curvature operator in the left hand side of (2.14). Indeed, (2.14) can be thought as a linear equation for u with coefficients defined by $a_{ij} = (1+|Du|^2)^{-1/2}\delta_{ij}$. Its uniform ellipticity is ensured only after establishing that u is Lipschitz. The same occurs regarding the ellipticity of the equation for the derivatives of u. This is equation (3.5) with $G(p) = (1 + |p|^2)^{1/2}$. These questions have to be attacked for each particular equation. See [33] for a gradient bound, as well as the blow-up technique and a crucial *monotonicity formula* for minimal surfaces.

Fully nonlinear equations. The previous variational techniques cannot be applied in the fully nonlinear context, since the equation cannot be integrated by parts in general. Another linear theory (for *nondivergence form equations with measurable coefficients*) is needed. We describe it in next sections. Its main tools are the ABP estimate (which plays the role of energy or Caccioppoli estimates in the variational theory) and the Krylov-Safonov Harnack inequality (which is the analogue of the De Giorgi-Nash-Moser theory). We will also describe the theory of viscosity solutions, which play for these equations the role of weak or energy solutions in the variational theory.

We emphasize that the variational theory is not a consequence of the fully nonlinear theory, and neither the other way around. They are independent theories which are useful in different situations.

4 The ABP technique. Applications to symmetry problems

During the sixties, Alexandroff, Bakelman, and Pucci introduced a method —which we call the ABP method— to prove the ABP estimate, Theorem 4.2 below. The ABP estimate is an L^∞ bound for solutions of uniformly elliptic equations $Lu = f(x)$ written in nondivergence form and with measurable coefficients. It plays a key role in the regularity theory for fully nonlinear elliptic equations, which is described in section 5.

In this section we first describe the ABP method with an application found by the author [10] that gives a new and simple proof of the classical isoperimetric problem. Later we show a very useful application of the ABP estimate to maximum principles and symmetry results.

4.1 Solution of the isoperimetric problem

The isoperimetric problem, already introduced in subsection 2.5, states:

Theorem 4.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$. Then,*

$$\frac{|\partial\Omega|}{|\Omega|^{\frac{n-1}{n}}} \geq \frac{|\partial B_1|}{|B_1|^{\frac{n-1}{n}}}, \tag{4.1}$$

where B_1 is the open unit ball of $\mathbb{R}^n$, $|\Omega|$ denotes the measure of Ω, and $|\partial\Omega|$ the perimeter of Ω. In addition, (4.1) *is an equality if and only if Ω is a ball.*

The following is a new and simple proof of the theorem found by the author. It uses the ABP method. It was only published previously in Catalan [8]. Further results in the same spirit will appear in [10].

Note that the proof applies to domains which are not necessarily convex. But we assume the domain to be regular enough. There are stronger versions of the isoperimetric problem in the sense that hold for larger classes of non smooth domains —see [4, 20].

Proof of Theorem 4.1. Let v be the solution of the following Neumann linear problem for the Laplacian:

$$\left\{ \begin{array}{rcll} \Delta v &=& |\partial\Omega|/|\Omega| & \text{in } \Omega \\ \partial v/\partial\nu &=& 1 & \text{on } \partial\Omega, \end{array} \right. \tag{4.2}$$

where $\partial v/\partial\nu$ is the exterior normal derivative of v. The constant $|\partial\Omega|/|\Omega|$ has been chosen such that the problem has a unique solution v up to an additive constant. This is a consequence of the relation $\int_\Omega \Delta v = \int_{\partial\Omega} \partial v/\partial\nu$, valid for all functions v, which gives a necessary and sufficient condition for the existence of solution. We have, moreover, that v is smooth in $\overline{\Omega}$.

Consider the *lower contact set* of v, defined by

$$\Gamma_v = \left\{ x \in \Omega \ : \ v(y) \geq v(x) + \nabla v(x) \cdot (y - x) \ \text{ for all } y \in \overline{\Omega} \right\}. \tag{4.3}$$

It is the set of points x such that the tangent hyperplane to the graph v at x lies below v in all of $\overline{\Omega}$. We claim that

$$B_1 \subset \nabla v(\Gamma_v), \tag{4.4}$$

where B_1 is the open unit ball of $\mathbb{R}^n$ centered at 0.

To prove (4.4), take $p \in \mathbb{R}^n$ with $|p| < 1$. Let $x \in \overline{\Omega}$ satisfy

$$\min_{y\in\overline{\Omega}} \{v(y) - p \cdot y\} = v(x) - p \cdot x. \tag{4.5}$$

Recall that, up to a sign, this is the *Legendre transform* of v. If we had that $x \in \partial\Omega$, then the exterior normal derivative of $v(y) - p \cdot y$ at x would be nonpositive, and therefore $(\partial v/\partial\nu)(x) \leq p \cdot \nu \leq |p| < 1$, a contradiction with (4.2). Thus $x \in \Omega$ and, as a consequence, x is an interior minimum of the function $v(y) - p \cdot y$. In particular, $p = \nabla v(x)$ and $x \in \Gamma_v$. We have proved statement (4.4).

It is interesting to visualize in a geometric way the proof of (4.4). Consider the graphs of the functions $p \cdot y + c$ where $c \in \mathbb{R}$. They are parallel hyperplanes that, for c close to $-\infty$, lie below the graph of v. We let c increase and consider the first constant c for which there is a contact with the graph of v at some point x. It is clear that the contact point $x \notin \partial\Omega$, since the "slope" of the hyperplane is less than 1 (because $|p| < 1$), while $\partial v/\partial\nu \equiv 1$ on $\partial\Omega$.

From (4.4) we deduce that

$$|B_1| \leq |\nabla v(\Gamma_v)| = \int_{\nabla v(\Gamma_v)} dp \leq \int_{\Gamma_v} \det D^2 v(x)\ dx. \tag{4.6}$$

The last inequality follows from the area formula (see [28]), an extension of the change of variables formula, valid for Lipschitz maps which are not necessarily bijective. We have applied this formula to the map $\nabla v : \Gamma_v \to \mathbb{R}^n$ and have also used that its Jacobian, $\det D^2 v$, is nonnegative in Γ_v, by the definition of this set.

The classical inequality between the geometric and arithmetic means, applied to the eigenvalues of $D^2 v(x)$ (which are nonnegative in Γ_v), gives that

$$\det D^2 v \leq \left(\frac{\Delta v}{n}\right)^n \quad \text{in } \Gamma_v. \tag{4.7}$$

Combining this inequality with (4.6) and $\Delta v \equiv |\partial\Omega|/|\Omega|$, we obtain

$$|B_1| \leq \left(\frac{|\partial\Omega|}{n|\Omega|}\right)^n |\Gamma_v| \leq \left(\frac{|\partial\Omega|}{n|\Omega|}\right)^n |\Omega|.$$

Since $|\partial B_1| = n|B_1|$, we conclude the isoperimetric inequality (4.1).

Recall that if $\Omega = B_1$ then $v(x) = |x|^2/2$ is the solution of problem (4.2) and, in particular, all the eigenvalue of $D^2 v(x)$ are equal. Therefore, all inclusions and inequalities in (4.4), (4.6), and (4.7) are equalities when $\Omega = B_1$. This explains why the proof gives the isoperimetric inequality with optimal constant.

The previous proof can also be used to show that balls are the only smooth domains for which equality occurs in (4.1) —see [8, 10]. □

4.2 ABP estimate. Maximum principle in small domains

For an application to the symmetry result of next subsection, we need to consider a slightly more general equation than (3.9), $a_{ij}(x)\partial_{ij} u = f(x)$. Throughout this subsection, L denotes an operator of the form

$$Lu = a_{ij}(x)\partial_{ij} u + b_i(x)\partial_i u + c(x)u,$$

in a bounded domain $\Omega \subset \mathbb{R}^n$. We assume that L is *uniformly elliptic* —i.e., L satisfies (3.10) and has bounded measurable coefficients:

$$\left(\Sigma b_i^2\right)^{1/2} \leq b \text{ and } |c| \leq \tilde{b} \quad \text{in } \Omega, \tag{4.8}$$

for some nonnegative constants b and $\tilde{b}$.

The following result, called ABP estimate, was proved by Alexandroff, Bakelman, and Pucci in the sixties. It is an essential tool in the regularity theory for fully nonlinear equations. In its statement, $W^{2,n}_{\mathrm{loc}}(\Omega)$ denotes the Sobolev space of functions that, together with their second derivatives, belong to $L^n_{\mathrm{loc}}(\Omega)$.

Theorem 4.2. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and that $c \leq 0$ in Ω. Let $u \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ satisfy $Lu \geq f$ in Ω and $u \leq 0$ on $\partial\Omega$, where $f \in L^n(\Omega)$. Then,*

$$\sup_{\Omega} u \leq C \operatorname{diam}(\Omega) \, \|f\|_{L^n(\Gamma^{u^+})},$$

where

$$\Gamma^{u^+} = \left\{x \in \Omega : u(x) > 0 \text{ and } u^+(y) \leq u(x) + \nabla u(x) \cdot (y - x) \text{ for all } y \in \overline{\Omega}\right\}$$

is the upper contact set of $u^+ := \max(u, 0)$, and C is a constant depending only on n, λ, and $b \cdot \operatorname{diam}(\Omega)$.

The fact that the L^n norm of f is computed in Γ^{u^+} will be useful in several occasions in section 5.

An improved version of the ABP estimate, where the factor $\operatorname{diam}(\Omega)$ is replaced by $|\Omega|^{1/n}$, was found by the author in [6].

Proof of Theorem 4.2. We use the same method of the proof of Theorem 4.1. We give the details of the proof when $b_i \equiv 0$ —such operators without first order terms are the ones that we will consider in next subsections. See section 9.1 of [32] for the general case $b_i \not\equiv 0$.

Let $M := \sup_\Omega u = u(x_0) > 0$ be achieved at $x_0 \in \Omega$ —recall that $u \leq 0$ on $\partial\Omega$. Let $d = \operatorname{diam}(\Omega)$. We work with the function $v := -u^+$. Hence, $-M = \inf_\Omega v = v(x_0)$, $v \leq 0$ in Ω, and $v \equiv 0$ on $\partial\Omega$.

Consider the lower contact set Γ_v of v, defined by (4.3). Note that in this set we have $v < 0$, and hence $u > 0$. Let $A(x) := [a_{ij}(x)]$ and note that

$$\operatorname{tr}(A(x)D^2 v) = -a_{ij}(x)\partial_{ij} u = -Lu + c(x)u \leq -Lu \leq -f(x) \qquad \text{in } \Gamma_v, \tag{4.9}$$

since $c \leq 0$.

Now the statement that replaces (4.4) is

$$B_{M/d} \subset \nabla v(\Gamma_v).$$

This is proved, as before, using the Legendre transform (4.5) of v and checking that the minimum is achieved at an interior point $x \in \Omega$ and not at the boundary $\partial\Omega$. Indeed, for every $y \in \partial\Omega$ we have that $v(y) - p \cdot y = -p \cdot y >$

$-M - p \cdot x_0 = v(x_0) - p \cdot x_0$, since $|y - x_0| \le d$ and we took p such that $|p| < M/d$.

Using the area formula we deduce that

$$c(n)(M/d)^n = |B_{M/d}| \le |\nabla v(\Gamma_v)| \le \int_{\Gamma_v} \det D^2 v(x)\, dx.$$

For $x \in \Gamma_v$, instead of proceeding as in (4.7), we now bound $\det D^2 v$ as follows. (4.9) leads to

$$\begin{aligned} \det D^2 v(x) &= \frac{1}{\det A(x)} \det(A(x) D^2 v(x)) \le \lambda^{-n} \det(A(x) D^2 v(x)) \\ &\le \lambda^{-n} \left\{ \frac{\operatorname{tr}(A(x) D^2 v(x))}{n} \right\}^n \le (n\lambda)^{-n} (-f(x))^n \le C|f(x)|^n. \end{aligned}$$

We deduce the ABP estimate, $\sup_\Omega u = M \le Cd \, \|f\|_{L^n(\Omega)}$. □

Next, we introduce a standard terminology in bounded domains.

Definition 4.3. We say that *the maximum principle holds for the operator L in Ω* if, whenever $u \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$,

$$Lu \ge 0 \text{ in } \Omega, \quad \text{and} \quad u \le 0 \text{ on } \partial\Omega,$$

then necessarily $u \le 0$ in Ω.

The following result is an immediate consequence of Theorem 4.2.

Corollary 4.4. *Let Ω be a bounded domain of $\mathbb{R}^n$. If $c \le 0$ in Ω, then the maximum principle holds for L in Ω.*

As we will see in next subsection on symmetry properties of solutions, the condition $c \le 0$ in Ω is, however, too restrictive for some applications. The following maximum principle in domains of small measure does not make any assumption on the sign of c and it will be very useful.

Proposition 4.5. *Assume that Ω is a bounded domain of $\mathbb{R}^n$. Then, there exists a constant $\delta > 0$ depending only on n, λ, b, $\tilde{b}$, and* $\operatorname{diam}(\Omega)$, *such that the maximum principle holds for L in Ω if the measure of Ω satisfies*

$$|\Omega| \le \delta.$$

This proposition is a consequence of the ABP estimate that was first noted by Bakelman, later by Varadhan, and then extensively used by Berestycki and Nirenberg (see next subsection). Similar maximum principles in small domains can be obtained also for divergence form operators $\mathcal{L}$ of the form (3.8). For this, one uses techniques from the variational theory which are clearly presented by Brezis in [5].

Proof of Proposition 4.5. Let u satisfy $Lu \geq 0$ in Ω and $u \leq 0$ on $\partial\Omega$. Let $d = \operatorname{diam}(\Omega)$ and $c = c^+ - c^-$, where $c^+ = \max(c, 0)$ and $c^- = \max(-c, 0)$. Consider the operator $L_0 = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i$. Writing $Lu \geq 0$ in the form

$$(L_0 - c^-)u \geq -c^+u \geq -c^+u^+,$$

we may apply the ABP estimate to the operator $L_0 - c^-$ and obtain

$$\begin{aligned} \sup_\Omega u &\leq C(n, \lambda, b, d)\|c^+u^+\|_{L^n(\Omega)} \\ &\leq C(n, \lambda, b, \tilde{b}, d)|\Omega|^{1/n} \sup_\Omega u^+. \end{aligned}$$

If $C(n, \lambda, b, \tilde{b}, d)|\Omega|^{1/n} \leq 1/2$, we conclude that $u \leq 0$ in Ω. □

4.3 Symmetry of solutions: the moving planes method

The goal of this section is to prove the following symmetry result of Berestycki and Nirenberg [3] for positive solutions of semilinear problems.

Theorem 4.6. *Let Ω be a bounded domain of $\mathbb{R}^n$ (not necessarily smooth) which is convex in the x_1-direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Let $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ be a solution of the problem*

$$\begin{cases} \Delta u + f(u) &= 0 \quad \text{in } \Omega \\ u &> 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Assume that f is Lipschitz.

Then, u is symmetric with respect to x_1, i.e., $u(x_1, y) = u(-x_1, y)$ for every $(x_1, y) \in \Omega$. Moreover, the partial derivative of u with respect to x_1 satisfies

$$u_{x_1} < 0 \quad \text{in } \Omega \cap \{x_1 > 0\}.$$

In particular, if $\Omega = B_R$ is a ball, then u is radially symmetric —that is, $u(x) = u(|x|)$ and $u_r < 0$ for $0 < r = |x| < R$.

When Ω is a smooth domain, this symmetry result was already proved in the classical paper of Gidas, Ni, and Nirenberg [29]. The proof given in [29] used a version of the maximum principle —the Hopf boundary lemma— which did not allow some domains Ω with corners, such as cubes. Next, we present the improved method of Berestycki and Nirenberg [3]. It replaces the use of the Hopf boundary lemma by the maximum principle in domains of small measure —Proposition 4.5 in the previous subsection. In this way, the proof applies to nonsmooth domains, such as cubes.

The proof of Theorem 4.6 uses a technique due to Alexandroff called the *moving planes method*. He developed this method to establish that every smooth closed hypersurface, embedded in $\mathbb{R}^n$ and of constant mean curvature, must be a sphere. Recall that this is a symmetry problem that appeared in subsection 2.5 on the isoperimetric problem.

Proof of Theorem 4.6. We denote points $x \in \mathbb{R}^n$ by $x = (x_1, y)$, $y \in \mathbb{R}^{n-1}$. It suffices to show

$$u(x_1, y) < u(x_1^*, y) \quad \text{if } -x_1 < x_1^* < x_1 \tag{4.10}$$

and

$$u_{x_1} < 0 \quad \text{if } x_1 > 0 \tag{4.11}$$

whenever $(x_1, y) \in \Omega$. Indeed, letting $x_1^* \to -x_1$ we get $u(x_1, y) \leq u(-x_1, y)$. The same result with the coordinate x_1 changed by $-x_1$ gives the symmetry: $u(x_1, y) = u(-x_1, y)$.

To show (4.10) and (4.11), we use the method of moving planes. Define $a = \sup_\Omega x_1$. For $0 < \lambda < a$, we consider the hyperplane T_λ and the set Σ_λ defined by

$$T_\lambda = \{x_1 = \lambda\} \quad \text{and} \quad \Sigma_\lambda = \{x \in \Omega : x_1 > \lambda\} \subset \Omega.$$

For $x \in \mathbb{R}^n$ we denote by

$$x^\lambda = (2\lambda - x_1, y)$$

the reflection of x with respect to T_λ. We consider the reflection of Σ_λ,

$$\Sigma'_\lambda = \left\{x^\lambda : x \in \Sigma_\lambda\right\} \subset \Omega,$$

which is contained in Ω, by the assumptions of the theorem on symmetry and convexity of Ω with respect to x_1. Hence, the function

$$w_\lambda(x) := u(x) - u(x^\lambda) \quad \text{for } x \in \overline{\Sigma}_\lambda$$

is well defined.

Since the Laplacian is invariant under reflections, the function $x \mapsto u(x^\lambda)$ satisfies the same semilinear equation $\Delta v + f(v) = 0$ as u. Thus, the difference w_λ satisfies the linear equation

$$\begin{aligned} 0 &= \Delta w_\lambda + f(u(x)) - f(u(x^\lambda)) \\ &= \Delta w_\lambda + c_\lambda(x) w_\lambda, \end{aligned}$$

where

$$c_\lambda(x) = \frac{f(u(x)) - f(u(x^\lambda))}{u(x) - u(x^\lambda)}$$

—we take $c_\lambda = 0$ if $u(x) = u(x^\lambda)$. Note that $\partial\Sigma_\lambda$ has two parts, one contained in T_λ and the other in $\partial\Omega$. Using that $u = 0$ on $\partial\Omega$ and $u > 0$ in Ω, we conclude

$$\left\{ \begin{aligned} \Delta w_\lambda + c_\lambda(x) w_\lambda &= 0 \quad \text{in } \Sigma_\lambda \\ w_\lambda &\leq 0 \quad \text{on } \partial\Sigma_\lambda, \quad w_\lambda \not\equiv 0. \end{aligned} \right. \tag{4.12}$$

Moreover, $|c_\lambda| \leq \tilde{b}$, where $\tilde{b}$ is the Lipschitz constant of f on $[0, \sup_\Omega u]$.

To prove (4.10) and (4.11) it suffices to verify

$$w_\lambda < 0 \quad \text{in } \Sigma_\lambda, \quad \text{for every } \lambda \in (0, a). \tag{4.13}$$

Indeed, it then follows from the Hopf lemma (see [32]) that on $T_\lambda \cap \Omega$, where $w_\lambda = 0$, we have $0 > (w_\lambda)_{x_1} = 2u_{x_1}$.

Now, if $a - \lambda$ is small then $\Sigma_\lambda \subset \Omega \cap \{\lambda < x_1 < a\}$, and hence Σ_λ has small measure. In particular, the maximum principle holds for the operator $\Delta + c_\lambda$ in Σ_λ if $a - \lambda$ is small, by Proposition 4.5. We deduce from (4.12) that $w_\lambda \le 0$ in Σ_λ. Now, the strong maximum principle (see [32]) gives that $w_\lambda < 0$ in Σ_λ. We have proved (4.13) for $a - \lambda$ small.

Let (λ_0, a) be the largest open interval of parameters for which (4.13) holds. We want to show that $\lambda_0 = 0$. We suppose $\lambda_0 > 0$ and show that it leads to contradiction. First, by continuity we have $w_{\lambda_0} \le 0$ in Σ_{λ_0} and, by the strong maximum principle, $w_{\lambda_0} < 0$ in Σ_{λ_0}.

Next, let $\delta > 0$ be the constant of Proposition 4.5. Let $K \subset \Sigma_{\lambda_0}$ be a compact set such that $|\Sigma_{\lambda_0} \setminus K| \le \delta/2$. We then have $w_{\lambda_0} \le -\eta < 0$ in K for some constant η, since K is compact. Hence, $w_{\lambda_0-\varepsilon} < 0$ in K and $|\Sigma_{\lambda_0-\varepsilon} \setminus K| \le \delta$ for $\varepsilon > 0$ small enough, by continuity.

We now apply the maximum principle in $\Sigma_{\lambda_0-\varepsilon} \setminus K$. We have

$$\begin{cases} \Delta w_{\lambda_0-\varepsilon} + c_{\lambda_0-\varepsilon}(x) w_{\lambda_0-\varepsilon} &= 0 \quad \text{in } \Sigma_{\lambda_0-\varepsilon} \setminus K \\ w_{\lambda_0-\varepsilon} &\le 0 \quad \text{on } \partial(\Sigma_{\lambda_0-\varepsilon} \setminus K). \end{cases}$$

Note that $\partial(\Sigma_{\lambda_0-\varepsilon} \setminus K)$ has one part contained in K, and we have used that $w_{\lambda_0-\varepsilon} < 0$ in K. Since $|\Sigma_{\lambda_0-\varepsilon} \setminus K| \le \delta$, Proposition 4.5 and the strong maximum principle give $w_{\lambda_0-\varepsilon} < 0$ in $\Sigma_{\lambda_0-\varepsilon} \setminus K$. Therefore, $w_{\lambda_0-\varepsilon} < 0$ in $\Sigma_{\lambda_0-\varepsilon}$, which contradicts the maximality of the interval (λ_0, a). □

The condition $u > 0$ in Ω in the previous theorem is, in general, necessary to conclude symmetry. As a simple example, there exist changing sign eigenfunctions of the Laplacian in a ball which are not radially symmetric.

We refer to [29, 3, 9] and references therein for symmetry results concerning more general equations, such as fully nonlinear equations $F(D^2u, Du, u, x) = 0$, and also more general domains, for instance some unbounded domains.

5 Harnack inequality. Fully nonlinear elliptic PDEs

To continue with applications of the ABP estimate, we start this section describing the Krylov-Safonov and the Evans regularity theories for fully nonlinear equations, giving the main ideas of all the crucial results. This is why we postpone to further subsections questions such as the general definition of uniformly elliptic fully nonlinear operator (which includes non differentiable operators), the examples from controlled diffusion processes, and also the considerations on viscosity solutions.

We consider the equation $F(D^2u, x) = 0$ for $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. We assume that $F = F(M, x)$ is C^1 with respect to $M \in \mathcal{S}$, where $\mathcal{S}$ is the space of $n \times n$ symmetric matrices. We denote its first order partial derivatives by $F_{ij} = \partial F / \partial m_{ij}$. We always assume that F is uniformly elliptic, i.e., for some constants $0 < \lambda \le \Lambda$ we have

$$\lambda|\xi|^2 \le F_{ij}(M,x)\xi_i\xi_j \le \Lambda|\xi|^2 \qquad \text{for all } M \in \mathcal{S}, x \in \Omega, \text{ and } \xi \in \mathbb{R}^n. \quad (5.1)$$

The conclusions of the theory state that solutions u are $C^{1,\alpha}$ for some small positive α, and that if in addition F is convex (or concave) in D^2u, then $u \in C^{2,\alpha}$ for some small positive α. The validity of $C^{2,\alpha}$ estimates, or even $C^{1,1}$ estimates, for nonconvex (and nonconcave) operators is still a major open problem —see [12].

In subsection 5.4, we will see that a perturbation result of Caffarelli allows (by a freezing coefficients technique) to reduce the problem of regularity for $F(D^2u, x) = 0$, to study the equation with constant coefficients

$$F(D^2u) = 0 \qquad \text{in } B_1 \subset \mathbb{R}^n.$$

We differentiate this equation with respect to the variable x_k. Writing $u_k = \partial_k u$, we obtain

$$Lu_k = a_{ij}(x)\partial_{ij}u_k := F_{ij}(D^2u(x))\,\partial_{ij}u_k = 0 \qquad \text{in } B_1, \quad (5.2)$$

which is a linear equation $Lu_k = 0$ for the derivative u_k. By (5.1), we know that L is uniformly elliptic, with ellipticity constants independent of the regularity of u. Note that making a regularity hypothesis on the coefficients $a_{ij}(x) = F_{ij}(D^2u(x))$ would mean to make a regularity hypothesis on the second derivatives of u. But this is what we want to prove, and hence we cannot assume. Therefore, we cannot use Schauder's theory (described in subsection 5.4). The only known way to continue consists on developing estimates (for uniformly elliptic linear equations in nondivergence form) independent of the modulus of continuity of the coefficients. This is the theory that we describe next.

5.1 Krylov-Safonov theory and $\mathbf{C^{1,\alpha}}$ regularity for $\mathbf{F(D^2u) = 0}$

In 1979 the theory experimented a substantial progress. Krylov and Safonov [44, 45] proved a deep result, the Harnack inequality for the operator L without any regularity hypothesis on the coefficients.

5.1.1 Local ABP estimate. Krylov-Safonov Harnack inequality

Throughout subsection 5.1.1, L is a uniformly elliptic operator of the form

$$L = a_{ij}(x)\partial_{ij}$$

with measurable coefficients. Hence it satisfies the uniform ellipticity condition (3.10) for some constants λ and Λ.

The statement of the Harnack inequality may be divided into two parts. The first one is simpler and applies to subsolutions. It is the following *local ABP estimate* —see also Theorem 9.20 of [32] and Theorem 4.8(2) of [14].

Proposition 5.1. *Let $u \in W^{2,n}(B_1)$ satisfy $Lu \geq f$ in B_1, where $f \in L^n(B_1)$. Then, for every $p > 0$, we have*

$$\sup_{B_{1/4}} u \leq C_p \left\{ \left(\int_{B_{1/2}} (u^+)^p \right)^{1/p} + \|f\|_{L^n(B_1)} \right\}, \tag{5.3}$$

where C_p is a constant depending only on n, λ, Λ, and p.

Sketch of the proof. Consider the function $v := (1 - |x|^2)^\beta u$. Since it vanishes on ∂B_1, we can apply to it the ABP estimate (Theorem 4.2) for the operator $a_{ij}\partial_{ij}$. Hence, we compute $a_{ij}\partial_{ij}v$. Its expression contains first order terms involving $\partial_i u$. We control them by a term involving the value of u and a term in ∇v.

To bound ∇v, we use that the integral on the right hand side of the ABP estimate is computed on the upper contact set Γ^{v^+}. By the concavity of v^+ on this set, one has $|\nabla v| \leq v/(1-|x|)$ in Γ^{v^+}. Finally, one chooses β appropriately to obtain the desired estimate. See section 9.7 of [32] for details. □

The second part of the Harnack inequality consists in a so called weak Harnack inequality for nonnegative supersolutions. The first step towards it is the following statement, which follows easily from the ABP estimate. Here $Q_r = (-r/2, r/2)^n$ denotes the cube centered at 0 with side length r.

Lemma 5.2. *There exist constants $\varepsilon_0 > 0, 0 < \mu < 1$, and $M > 1$, depending only on n, λ, and Λ, for which the following assertion holds. Let $u \in W^{2,n}(Q_{4\sqrt{n}})$ satisfy $Lu \leq f$ in $Q_{4\sqrt{n}}$,*

$$u \geq 0 \text{ in } Q_{4\sqrt{n}}, \quad \inf_{Q_3} u \leq 1, \quad \text{and} \quad \|f\|_{L^n(Q_{4\sqrt{n}})} \leq \varepsilon_0.$$

Then,

$$|\{u > M\} \cap Q_1| \leq (1 - \mu)|Q_1|.$$

Proof. This is Lemma 4.5 of [14], where all details can be found. The idea is to apply the ABP estimate to $w = -u - \varphi$, where φ is a radial function satisfying $\varphi \leq -2$ in Q_3, $\varphi \geq 0$ outside $B_{2\sqrt{n}}$, and $L\varphi \leq 0$ outside Q_1. One can prove easily the existence of such function, satisfying also some other easier bounds that we will need through the proof.

Note that $w \leq 0$ on $\partial B_{2\sqrt{n}}$ and that $\sup_{Q_3} w \geq 1$. Applying the ABP estimate to w in $B_{2\sqrt{n}}$, we get

$$1 \leq C \left(\int_{\Gamma^{w^+}} (|f| + (L\varphi)^+)^n \right)^{1/n} \leq C\|f\|_{L^n(Q_{4\sqrt{n}})} + C|\Gamma^{w^+} \cap Q_1|^{1/n}.$$

Taking ε_0 small enough, we deduce

$$\frac{1}{2} \leq C|\Gamma^{w^+} \cap Q_1|^{1/n} \leq C|\{u \leq M\} \cap Q_1|^{1/n}.$$

We have used that $w > 0$ in Γ^{w^+}, and hence $u \le -\varphi(x) \le M$ in this set. $\square$

Next, with u as in Lemma 5.2, one considers the distribution function of u in Q_1, defined by $\lambda_u(t) := |\{u > t\} \cap Q_1|$. One applies the rescaled version of Lemma 5.2 to u/M^k at every scale of space (that is, in small cubes). This combined with a version of the *Calderón-Zygmund cube decomposition* (see Lemma 4.2 of [14]), leads to power decay for λ_u, i.e.,

$$\lambda_u(t) \le C t^{-\varepsilon},$$

for some $\varepsilon > 0$ depending only on n, λ, and Λ. This bound, rescaled and applied together with a simple covering by balls to pass from cubes to balls, immediately gives the important *weak Harnack inequality* of Krylov-Safonov for nonnegative supersolutions:

Theorem 5.3. *Let $u \in W^{2,n}(B_1)$ satisfy $u \ge 0$ and $Lu \le f$ in B_1, where $f \in L^n(B_1)$. Then,*

$$\left(\int_{B_{1/4}} u^{p_0} \right)^{1/p_0} \le C \left\{ \inf_{B_{1/2}} u + \|f\|_{L^n(B_1)} \right\},$$

where $p_0 > 0$ and C are constants depending only on n, λ, and Λ.

Proposition 5.1 and Theorem 5.3 (rescaled and applied together with a simple covering by balls) leads to the *Krylov-Safonov Harnack inequality*:

Theorem 5.4. *Let $u \in W^{2,n}(B_R)$ satisfy $u \ge 0$ and $Lu = f$ in B_R, where $f \in L^n(B_R)$. Then,*

$$\sup_{B_{R/2}} u \le C \left\{ \inf_{B_{R/2}} u + R\,\|f\|_{L^n(B_R)} \right\},$$

where C is a constant depending only on n, λ, and Λ.

The following are two important consequences of the Harnack inequality. First, a *Liouville theorem*, stating that bounded solutions of $Lu = 0$ in all of $\mathbb{R}^n$ are constant —see [14, 32] for the proof, and also subsection 5.5 for an extension to manifolds. Second, the Hölder continuity of solutions of $Lu = f$:

Corollary 5.5. *Let $u \in W^{2,n}(B_1)$ satisfy $Lu = f$ in B_1, where $f \in L^n(B_1)$. Then $u \in C^\alpha(\overline{B}_{1/2})$ and*

$$\|u\|_{C^\alpha(\overline{B}_{1/2})} \le C \left\{ \|u\|_{L^\infty(B_1)} + \|f\|_{L^n(B_1)} \right\},$$

where $0 < \alpha < 1$ and C are positive constants depending only on n, λ, and Λ.

Proof. For $0 < r < 1$, define $M_r = \sup_{B_r} u$, $m_r = \inf_{B_r} u$, and $o_r = M_r - m_r$. The quantity o_r is called the oscillation of u in B_r.

Applying Theorem 5.4 to $u - m_r \geq 0$ and $M_r - u \geq 0$ in B_r, we obtain

$$M_{r/2} - m_r \leq C \left\{ m_{r/2} - m_r + r \, \|f\|_{L^n(B_r)} \right\}$$

and

$$M_r - m_{r/2} \leq C \left\{ M_r - M_{r/2} + r \, \|f\|_{L^n(B_r)} \right\}.$$

Adding these inequalities, we deduce

$$o_r + o_{r/2} \leq C \left\{ o_r - o_{r/2} + 2r \, \|f\|_{L^n(B_r)} \right\}$$

and as a consequence,

$$o_{r/2} \leq \frac{C-1}{C+1} o_r + \frac{C}{C+1} 2r \, \|f\|_{L^n(B_r)} \leq (1-\delta) \left\{ o_r + 2r \, \|f\|_{L^n(B_r)} \right\}, \quad (5.4)$$

with $0 < \delta < 1$ depending only on n, λ, and Λ.

Applying repeatedly this inequality, in balls of radius $r = 1/2^i$ centered at a given point x_0, we observe that the oscillation of u around x_0 decreases as a power of the radius. This is the same as saying that u is C^α at the point x_0, for a certain α depending only on n, λ, and Λ. □

Remark 5.6. From (5.4) with $f \equiv 0$, we also deduce the following important property of solutions of second order homogeneous elliptic equations. If $Lu = 0$, then the oscillation of u in a ball of radius $r/2$ is reduced by a multiplicative factor less than 1 with respect to the oscillation in the concentric ball of radius r.

This property, since it holds at every scale by rescale invariance of the equation, becomes equivalent to the Hölder continuity of the solution.

5.1.2 $\mathbf{C^{1,\alpha}}$ regularity for $\mathbf{F(D^2u) = 0}$

We can now present an important consequence of the Krylov-Safonov theory. If $F \in C^1$ and $F(D^2u) = 0$, then every derivative $u_k = \partial_k u$ is a solution of the uniformly elliptic linear equation (5.2). Since Corollary 5.5 makes no hypothesis on the regularity of the coefficients of the linear equation, we obtain a C^α estimate for $\partial_k u$, which reads

$$\|\partial_k u\|_{C^\alpha(\overline{B}_{1/2})} \leq C \|\partial_k u\|_{L^\infty(B_1)}.$$

As a consequence, we have a $C^{1,\alpha}$ estimate for u,

$$\|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \leq C \|u\|_{C^1(\overline{B}_1)}.$$

This estimate can be improved by replacing $\|u\|_{C^1(\overline{B}_1)}$ by $\|u\|_{L^\infty(B_1)} + |F(0)|$, as we state in the next theorem —see [11, 14, 24].

We have made the hypothesis $F \in C^1$ to differentiate $F(D^2u) = 0$ with respect to x_k, but the incremental quotients technique allows to obtain the same result for all F uniformly elliptic (in the sense of Definition 5.11 below), not necessarily C^1.

Theorem 5.7. *Let F be uniformly elliptic and $u \in C^2(B_1)$ be a solution of $F(D^2u) = 0$ in B_1. Then*

$$\|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \le C \left\{ \|u\|_{L^\infty(B_1)} + |F(0)| \right\},$$

where $0 < \alpha < 1$ and C are constants depending only on n, λ, and Λ.

5.2 $\mathbf{C^{2,\alpha}}$ regularity for concave or convex equations $\mathbf{F(D^2u) = 0}$

Up to this point we have differentiated the equation $F(D^2u) = 0$ once with respect to a given direction $e \in \mathbb{R}^n$, $|e| = 1$, obtaining $F_{ij}(D^2u(x))\partial_{ij}u_e = 0$, where $u_e = \partial_e u$. Now we assume that $F \in C^2$ is concave, we differentiate again with respect to e, and denote $u_{ee} = \partial_{ee}u$. Assuming that $u \in C^4$ and using the concavity of F, we deduce

$$\begin{aligned} 0 &= F_{ij}(D^2u(x))\partial_{ij}u_{ee} + F_{ij,rs}(D^2u(x))(\partial_{ij}u_e)(\partial_{rs}u_e) \\ &\le F_{ij}(D^2u(x))\partial_{ij}u_{ee} =: a_{ij}(x)\partial_{ij}u_{ee}. \end{aligned} \tag{5.5}$$

Therefore, every second pure derivative u_{ee} of u is a *subsolution* of a uniformly elliptic equation. In 1982, Evans [23] and Krylov [41, 42] used this fact to obtain a C^α estimate for D^2u. That is, they obtained the desired $C^{2,\alpha}$ estimate for u. We give a proof of this below, following an idea of Caffarelli.

Theorem 5.8. *Let F be uniformly elliptic and either concave or convex. If $u \in C^2(B_1)$ is a solution of $F(D^2u) = 0$ in B_1, then $u \in C^{2,\alpha}(B_1)$ and*

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \le C \left\{ \|u\|_{L^\infty(B_1)} + |F(0)| \right\},$$

where $0 < \alpha < 1$ and C are constants depending only on n, λ, and Λ.

It is still a major open problem to know if the hypothesis F concave (or convex) is needed in this theorem —see [12, 16, 53] for some results in this topic. Note that the hypothesis that F is concave can be replaced by F convex, since we can write the equation in the form $-F(-D^2(-u)) = 0$, that is, $G(D^2v) = 0$ where $v := -u$ and $G(M) := -F(-M)$. The new operator G is uniformly elliptic, and G is convex if F is concave.

Theorem 5.8 applies to *Bellman equations* (which are concave), but does not apply to *Isaacs equations*. These equations are defined below, in subsection 5.3.2. The $C^{1,\alpha}$ regularity of Theorem 5.7 applies to both of these types of equations.

The hypotheses $F \in C^2$ and $u \in C^4$, made in the beginning of this subsection to differentiate twice the equation, can be removed (see [14] for details). Thus, Theorem 5.8 applies to non differentiable F which are uniformly elliptic in the sense of Definition 5.11 below —such as Bellman and Isaacs operators, which are non differentiable in general.

There are versions of Theorems 5.7 and 5.8 which also apply to viscosity solutions. This is a class of weak solutions described below in subsection 5.3.3. Such theorems establish their $C^{1,\alpha}$and $C^{2,\alpha}$ regularity, respectively.

Theorems 5.7 and 5.8 do not apply directly to Monge-Ampère type equations (see the examples found in subsections 2.6 and 2.7). Indeed, as we saw in subsection 3.2, for these equations one needs to first prove the uniform convexity of the solution to ensure the uniform ellipticity of the operators. Thus, there is an additional nontrivial task to be carried out (see [32, 35]).

Next, we present a quite simple proof of the $C^{2,\alpha}$ estimate of Theorem 5.8 which was found by Caffarelli. It is based on a clever application of the weak Harnack inequality of Theorem 5.3 to the supersolutions $C - u_{ee}$. First, we need the following important fact.

Remark 5.9. Let F be uniformly elliptic —here the concavity of F is not needed. If M_1 and M_2 are symmetric matrices with $F(M_1) = F(M_2) = 0$, then

$$c_0\|M_2 - M_1\| \leq \|(M_2 - M_1)^+\| = \sup_{e\in\mathbb{R}^n, |e|=1} (e^t(M_2 - M_1)e)^+, \qquad (5.6)$$

where c_0 is a positive constant depending only on n, λ, and Λ. Here $\|\cdot\|$ denotes the $l^2 - l^2$ norm of a matrix. Since $M_2 - M_1$ is a symmetric matrix, it diagonalizes in an orthonormal basis. Let us call E such diagonal matrix containing all the eigenvalues $e_1, \cdots, e_n$ of $M_2 - M_1$. It is now clear that $\|M_2 - M_1\|$ (respectively, $\|(M_2 - M_1)^+\|$) is equal to the maximum of the values $|e_i|$ (respectively, $(e_i)^+$).

To check (5.6), note that

$$\begin{aligned} 0 &= [F((1-t)M_1 + tM_2)]_{t=0}^1 = \left[\int_0^1 F_{ij}((1-t)M_1 + tM_2)\,dt\right](M_2 - M_1)_{ij} \\ &=: \tilde{a}_{ij}(M_2 - M_1)_{ij}, \end{aligned}$$

where $\tilde{a}_{ij}$ is a positive definite symmetric matrix with all eigenvalues in $[\lambda, \Lambda]$, by (5.1). From this elliptic relation for $M_2 - M_1$, one easily concludes (5.6). Indeed, let $\tilde{A} = [\tilde{a}_{ij}]$ and E be the diagonal form of $M_2 - M_1$ as above. We have

$$\begin{aligned} 0 &= \tilde{a}_{ij}(M_2 - M_1)_{ij} = \operatorname{tr}(\tilde{A}(M_2 - M_1)) \\ &= \operatorname{tr}(\tilde{B}E) = \tilde{b}_{11}e_1 + \cdots + \tilde{b}_{nn}e_n, \end{aligned}$$

since the trace is invariant under change of basis. Recall that $\tilde{A}$, and hence also $\tilde{B} = [\tilde{b}_{ij}]$, are symmetric matrices with all eigenvalues in $[\lambda, \Lambda]$. From the last linear relation for the e_i, the claim follows.

Note that exactly the same argument gives that if u and v are two solutions of $F(D^2w) = 0$, then the difference $v - u$ satisfies a linear uniformly elliptic equation $\tilde{a}_{ij}(x)\partial_{ij}(v - u) = 0$ with measurable coefficients.

The following is the main lemma, due to Caffarelli, towards the Evans-Krylov theorem. It concerns the sets $D^2(B_1)$ and $D^2(B_{1/2})$.

Lemma 5.10. *Under the hypothesis of Theorem* 5.8, *assume that*

$$1 < \mathrm{diam} D^2u(B_1) \leq 2,$$

and that $D^2u(B_1)$ *is covered by* N *balls* $B^1, \ldots, B^N$ *of radius* ε *(in the space* $\mathcal{S}$ *of symmetric matrices), where* $N \geq 1$.

There exists $\varepsilon_0 > 0$ *depending only on* n, λ, *and* Λ, *such that if* $\varepsilon \leq \varepsilon_0$, *then* $D^2u(B_{1/2})$ *is covered by* $N-1$ *balls among* $B^1, \ldots, B^N$.

Proof. For $i = 1, \ldots, N$, we take $x_i \in B_1$ such that $B^i \subset B_{2\varepsilon}(M_i)$, where

$$M_i = D^2u(x_i).$$

Taking ε_0 such that $2\varepsilon \leq 2\varepsilon_0 \leq c_0/16$, we have that

$$B_{c_0/16}(M_1), \ldots, B_{c_0/16}(M_N)$$

cover $D^2u(B_1)$. Since $D^2u(B_1)$ has diameter at most 2, every M_i belongs to one closed ball $\bar{B}$ of radius 2 in $\mathcal{S}$. Let N' be the maximum number of points in the ball $\bar{B}$ such that the distance between any two of them is at least $c_0/16$, where c_0 is the constant in (5.6) . We have that N' depends only on n and c_0. Hence, we can assume that $\left\{B_{c_0/8}(M_i)\right\}_{i=1}^{N'}$ cover $D^2u(B_1)$, where $N' \leq N$ and N' (as all constants below) depends only on n, λ, and Λ.

It follows that $\left\{(D^2u)^{-1}B_{c_0/8}(M_i)\right\}_{i=1}^{N'}$ cover B_1 and, therefore, there exists one M_i, say M_1, such that

$$|(D^2u)^{-1}(B_{c_0/8}(M_1)) \cap B_{1/4}| \geq \eta > 0, \tag{5.7}$$

where η depends only on n, λ, and Λ.

Recall that $\mathrm{diam} D^2u(B_1) > 1$ and take $2\varepsilon \leq 2\varepsilon_0 \leq 1/4$. Since $\{B_{2\varepsilon}(M_i)\}_{i=1}^{N}$ cover $D^2u(B_1)$, it follows that there is one M_i, say M_2, such that $\|M_2 - M_1\| \geq 1/4$. Now, (5.6) gives the existence of $e \in \mathbb{R}^n$ with $|e| = 1$ such that

$$u_{ee}(x_2) \geq u_{ee}(x_1) + c_0/4. \tag{5.8}$$

Define

$$K = \sup_{B_1} u_{ee} \qquad \text{and} \qquad v = K - u_{ee}.$$

By (5.5), we have that $v \geq 0$ is a supersolution of a linear uniformly elliptic equation in B_1 with right hand side 0. (5.7) and (5.8) give that $|\{v \geq c_0/8\} \cap B_{1/4}| \geq \eta$. We can now apply Theorem 5.3 to v and get

$$\inf_{B_{1/2}} (K - u_{ee}) \geq C_1 > 0. \tag{5.9}$$

By the definition of K, and again since $\{B_{2\varepsilon}(M_i)\}_{i=1}^{N}$ cover $D^2u(B_1)$, there exists one j, with $1 \leq j \leq N$, such that

$$K - u_{ee}(x_j) < 3\varepsilon. \tag{5.10}$$

If we finally take $5\varepsilon \leq 5\varepsilon_0 \leq C_1$, then (5.9) and (5.10) lead to $D^2u(B_{1/2}) \cap B_{2\varepsilon}(M_j) = \emptyset$. Hence $D^2u(B_{1/2}) \cap B^j = \emptyset$ and the lemma follows. □

Lemma 5.10 is applied repeatedly in B_1, first to u, then to $w(x) := 4u(x/2)$, etc. Note that the initial number N of balls of radius ε_0 needed depends only on n, λ, and Λ (see the argument in the beginning of the previous proof). After repeated application of the lemma, and since we cannot run out of balls, it follows that $\operatorname{diam} D^2u(B_{1/2^N}) \leq 1$ —under the initial hypothesis that $\operatorname{diam} D^2u(B_1) = 2$. Hence the "oscillation" of D^2u decreases. This leads to the $C^{2,\alpha}$ estimate for u —as in Remark 5.6.

5.3 Fully nonlinear elliptic operators. Controlled diffusion processes

5.3.1 Definition of uniform ellipticity. Maximum principle

We consider equations of the form

$$F(D^2u(x), x) = 0, \tag{5.11}$$

for $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. The operator $F = F(M, x)$ is a real function defined in $\mathcal{S} \times \Omega$, where $\mathcal{S}$ denotes the space of $n \times n$ real symmetric matrices. To develop a complete regularity theory it is essential to assume uniformly ellipticity.

Definition 5.11. We say that F is *uniformly elliptic* if there exist two positive constants $0 < \lambda \leq \Lambda$ (called ellipticity constants) such that

$$\lambda\|N\| \leq F(M + N, x) - F(M, x) \leq \Lambda\|N\| \qquad \text{for all } N \geq 0, \tag{5.12}$$

for all matrix $M \in \mathcal{S}$, and for all $x \in \Omega$. Here $N \geq 0$ means that N is a nonnegative definite symmetric matrix, and $\|N\|$ is its highest eigenvalue.

Condition (5.12) implies that F is an increasing function of $M \in \mathcal{S}$. That is,

$$\text{if } \ M_1 < M_2 \text{ then } \ F(M_1, x) < F(M_2, x) \qquad \text{for all } x \in \Omega. \tag{5.13}$$

Here we have considered the usual order in $\mathcal{S}$, $M_1 < M_2$ if $M_2 - M_1$ is positive definite.

The first example of uniformly elliptic operator is obviously the Laplacian, and more generally every uniformly elliptic linear operator $Lu = a_{ij}(x)\partial_{ij}u$ satisfying the uniform ellipticity condition (3.10).

In next subsection we introduce Bellman and Isaacs operators, which are uniformly elliptic operators (and truly nonlinear). Note that the definition of uniform elliptic operator allows F to be non differentiable with respect to the variable M. This is an interesting point, since we will see that Bellman and Isaacs operators are usually non differentiable.

At the same time, note that for operators of class C^1, uniform ellipticity is equivalent (up to a change of the ellipticity constants) to condition (5.1) for the first derivatives $F_{ij}(M, x) = \frac{\partial F}{\partial m_{ij}}(M, x)$.

A consequence of ellipticity is the *maximum principle*:

Proposition 5.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $F(D^2u(x), x) \geq F(D^2v(x), x)$ in Ω. Then,*

$$\sup_{\Omega}(u - v) \leq \sup_{\partial\Omega}(u - v).$$

In particular, if we assume $F(0, x) = 0$ in Ω and $F(D^2u(x), x) \geq 0$ in Ω, then u attains its maximum on the boundary $\partial\Omega$.

Proof. If we had that $(u - v)(x_0) > \sup_{\partial\Omega}(u - v)$ for some $x_0 \in \Omega$, then $w(x) := u(x) - v(x) + \varepsilon|x - x_0|^2/2$ would also have an interior maximum $x_1 \in \Omega$ for a small enough ε (since $w(x_0) > \sup_{\partial\Omega} w$ for small ε). Therefore, we would have $D^2w(x_1) \leq 0$, that is, $D^2u(x_1) \leq D^2v(x_1) - \varepsilon\mathrm{Id} < D^2v(x_1)$. Now, (5.13) leads to $F(D^2u(x_1), x_1) < F(D^2v(x_1), x_1)$, a contradiction with the hypothesis.

In an alternative way, Proposition 5.12 also follows from the ABP estimate (Theorem 4.2) applied to a linear inequality satisfied by $u-v$ (which is obtained as in Remark 5.9). □

The maximum principle has a direct consequence: the *uniqueness* of the solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ (if it exists) for the Dirichlet problem

$$\begin{cases} F(D^2u, x) &= 0 \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega. \end{cases} \tag{5.14}$$

5.3.2 Controlled diffusion processes: Bellman and Isaacs equations

Bellman and Isaacs equations are uniformly elliptic fully nonlinear equations that appear in problems of stochastic optimization. We will introduce them next, formalizing in this way the simple relations already seen in subsection 2.1 between elliptic operators —such as (2.4), (2.6), and (2.8)— and random walks.

The state X_t of a system is given by the solution of a stochastic differential equation that depends on a certain control β, where $\beta = \beta_t \in \mathcal{B}$ is a stochastic process and $\mathcal{B}$ is a compact metric space. We want to keep the state X_t of the system inside an open region Ω in $\mathbb{R}^n$. The process X_t will stop at the exit time τ of the region $\overline{\Omega}$. We consider the cost function

$$\mathcal{J}(x, \beta) = E \int_0^\tau f_\beta(X_t)\, dt, \tag{5.15}$$

where $x = X_0$ is the initial state.

The goal is to minimize the cost $\mathcal{J}$ among all possible controls β, and thus we consider the *optimal cost function*

$$u(x) = \inf_{\beta \in \mathcal{B}} \mathcal{J}(x, \beta) \qquad \text{for } x \in \overline{\Omega}.$$

The dynamic programing principle establishes that u is the solution of the *Bellman equation*

$$F(D^2u, x) := \inf_{\beta \in \mathcal{B}} \{L_\beta u(x) + f_\beta(x)\} = 0, \tag{5.16}$$

with zero Dirichlet boundary conditions. Here, for each $\beta \in \mathcal{B}$, $L_\beta u = a_{ij}^\beta(x)\partial_{ij}u$ is a linear elliptic operator. The coefficients a_{ij}^β can be computed from the coefficients of the stochastic differential equation (see [26, 40] for more details).

We can relate the above considerations with the elementary approach to random walks in subsection 2.1. Assume that there is no control, that is, there is only one operator L_β and that it is $\frac{1}{2}\Delta$. Take $f_\beta \equiv 1$ in (5.15), so that $u(x)$ becomes the expected time to hit $\partial\Omega$ starting at x. By (5.16) (and its boundary conditions), u solves

$$\begin{cases} -\frac{1}{2}\Delta u &= 1 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.17}$$

which is problem (2.6) found in subsection 2.1 through an elementary (heuristic) way using the discretization of random walks.

On the other hand, stochastic differential games (see [47]) lead to *Isaacs equations*

$$F(D^2u, x) := \sup_{\gamma\in\mathcal{G}} \inf_{\beta\in\mathcal{B}} \{L_{\beta\gamma}\, u(x) + f_{\beta\gamma}(x)\} = 0. \tag{5.18}$$

If L_β and $L_{\beta\gamma}$ are all uniformly elliptic with constants λ and Λ independent of x, β, and γ, then the corresponding Bellman and Isaacs operators are uniformly elliptic, since condition (5.12) is stable under infima and suprema.

There is an important difference between these two families of operators. Being the infimum of linear operators, the Bellman operator is concave in the $D^2u = M$ variable. Its viscosity solutions (defined next) therefore enjoy $C^{2,\alpha}$ regularity. However, Isaacs operators are, in general, neither concave nor convex. As we said before, the validity of $C^{2,\alpha}$ (or even $C^{1,1}$) estimates for these equations is a major open problem. In this direction, Caffarelli and the author [12] have established a $C^{2,\alpha}$ regularity result for a particular type of Isaacs equations.

5.3.3 Viscosity solutions

In the early eighties, Crandall-Lions [21] and Evans [22] developed a theory of weak solutions (called viscosity solutions) for fully nonlinear equations. The idea consists on taking the maximum principle as a definition of solution.

Definition 5.13. Let u be a continuous function in Ω. We say that u is a *viscosity subsolution* of $F(D^2u, x) = 0$ in Ω (or that u satisfies $F(D^2u, x) \geq 0$ in Ω in the viscosity sense) when the following condition holds. If $x_0 \in \Omega$, $\phi \in C^2(\Omega)$, and $u - \phi$ has a local maximum at x_0, then

$$F(D^2\phi(x_0), x_0) \geq 0. \tag{5.19}$$

The definition of viscosity supersolution is the same if "local maximum" is replaced by "local minimum" and if $\geq$ is changed by $\leq$ in (5.19). We say that u is a viscosity solution when it is both viscosity subsolution and supersolution. Any classical solution $u \in C^2(\Omega)$ is clearly a viscosity solution, by the monotonicity of F.

Viscosity solutions are very useful due to their good stability and compacity properties (see chapter 2 of [14] for a fast introduction). Indeed, viscosity solutions provide a general *existence and uniqueness theory*. Recall that we have easily shown the uniqueness of classical solution to the Dirichlet problem (5.14). The uniqueness of viscosity solution for this problem is, however, a delicate question proved in 1988 by Jensen (see [24] for a nice introduction to this, and [14] for details). The existence of viscosity solution for the Dirichlet problem was proved by Ishii using the Perron method and the uniqueness result.

Hence, an existence and uniqueness theory for viscosity solutions of (5.14) is available, even in the case of operators F that are neither concave nor convex. This is very interesting since $C^{2,\alpha}$ estimates are not available for such operators.

5.4 Schauder, Calderón-Zygmund, and fully nonlinear extensions

For a solution of a second order elliptic equation one expects, in general, to control the second derivatives of the solution by the oscillation of the solution itself, as in Theorem 5.8.

For linear equations, the following $C^{2,\alpha}$ and $W^{2,p}$ interior a priori estimates hold. Let u be a solution of a linear uniformly elliptic equation of the form

$$a_{ij}(x)\partial_{ij}u = f(x) \quad \text{in } B_1 \subset \mathbb{R}^n. \tag{5.20}$$

Then, we have:

(a) *Schauder estimates*: if a_{ij} and f belong to $C^\alpha(\overline{B}_1)$, for some $0 < \alpha < 1$, then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and $\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(\overline{B}_1)})$, where C depends on the ellipticity constants and the $C^\alpha(\overline{B}_1)$-norm of a_{ij} —see [43] and Chapter 6 of [32].

(b) *Calderón-Zygmund estimates*: if $a_{ij} \in C(\overline{B}_1)$ and $f \in L^p(B_1)$, for some $1 < p < \infty$, then $u \in W^{2,p}(B_{1/2})$ and $\|u\|_{W^{2,p}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)})$, where C depends on the ellipticity constants and the modulus of continuity of the coefficients a_{ij} —see Chapter 9 of [32].

These statements should be understood as regularity results for appropriate linear small perturbations of the Laplacian. Indeed, these estimates are proved by regarding the equation $a_{ij}(x)\partial_{ij}u = f(x)$ as

$$a_{ij}(x_0)\partial_{ij}u = [a_{ij}(x_0) - a_{ij}(x)]\,\partial_{ij}u + f(x).$$

One applies to this equation the corresponding second derivatives estimates for the constant coefficients operator $a_{ij}(x_0)\partial_{ij}$ (that one can think of as the Laplacian), observing that the factor in the right hand side $a_{ij}(x_0) - a_{ij}(x)$ is small (locally around x_0) in some appropriate norm, due to the regularity assumptions made on a_{ij}.

Thus, the key point is to prove $C^{2,\alpha}$ and $W^{2,p}$ estimates for Poisson's equation

$$\Delta u = f(x).$$

Such estimates follow from an estimate for the Newtonian potential v of the function f, and an estimate for the harmonic function $u-v$. The $W^{2,p}$ estimate for the Newtonian potential of f is obtained through the Calderón-Zygmund cube decomposition technique for singular integrals (see chapter 9 of [32] for a clear exposition).

The goal is to extend this regularity results to the fully nonlinear case, that we will write in the form $F(D^2u, x) = f(x)$ by analogy with (5.20). The previous discussion shows that one should start considering the case of equations with constant "coefficients" $F(D^2u) = f(x)$ (here, we think of $F(D^2u)$ as being equal to $F(D^2u(x), x_0)$ for a fixed x_0). In fact, the key ideas already appear by considering the simpler equation

$$F(D^2u) = 0,$$

for which $C^{2,\alpha}$ regularity was discussed in subsections 5.1 and 5.2.

In 1989, Caffarelli [13] extended the perturbation technique described above in the linear case to the context of fully nonlinear equations $F(D^2u, x) = f(x)$. Under the assumptions of concavity of F in the variable M and enough regularity of F in the variable x, he established the following results (see [14]). If $f \in L^p$ and $n < p < \infty$ then $u \in W^{2,p}$ in the interior and there is a $W^{2,p}$ estimate for u (that is, an L^p estimate for the second derivatives of u). If $f \in C^\alpha$, where $0 < \alpha < 1$ depends on the ellipticity constants, then $u \in C^{2,\alpha}$ in the interior.

There exists also a $C^{2,\alpha}$ estimate up to the boundary for solutions of concave equations $F(D^2u) = 0$. This result was proved independently by Caffarelli-Nirenberg-Spruck and by Krylov in 1984, and leads to a theorem on existence of classical solutions to the Dirichlet problem (see [14, 32]).

5.5 Elliptic PDEs and optimal maps on Riemannian manifolds

The ABP estimate and the Euclidean theory of Krylov-Safonov have been extended by the author in [7] to equations on Riemannian manifolds with nonnegative curvature tensor. The main innovation of this work consists on finding appropriate replacements for the affine functions of Euclidean space ($p \cdot x + c$, $x \in \mathbb{R}^n$) which are used in the proof of the classical ABP estimate. Note that there is no corresponding notion of such functions when $x \in M$ and M is a manifold. This problem is solved in [7] by replacing hyperplanes or Euclidean affine functions by paraboloids, which have the functions "squared distance to a point" as analogues on manifolds.

That is, in $\Omega \subset \mathbb{R}^n$ we consider

$$\min_{y \in \overline{\Omega}} \left\{ v(y) + |y-p|^2/2 \right\} = v(x) + |x-p|^2/2 \tag{5.21}$$

instead of (4.5), and in a manifold $\Omega \subset M$ we consider

$$\min_{y \in \overline{\Omega}} \left\{ v(y) + d(y,p)^2/2 \right\} = v(x) + d(x,p)^2/2, \tag{5.22}$$

where d is the Riemannian distance in M. The equality satisfied at a minimum point of (4.5) was $p = \nabla v(x)$, which does not have a clear meaning when $v : \Omega \subset M \to \mathbb{R}$ (as the vectors $\nabla v(x)$ lie, when x varies, in different tangent spaces). On the other hand, minimum points of (5.21) satisfy

$$p = x + \nabla v(x),$$

which becomes, for minimum points of (5.22) on a manifold,

$$p = \exp_x \nabla v(x) \in M, \tag{5.23}$$

where $\exp_x$ stands for the exponential map with base at the point $x \in M$.

The area formula on a manifold is used to proceed with the ABP technique. Here, to control the Jacobian of the map $p = p(x)$ given by (5.23) one uses a lower bound on the sectional curvature of M. Assuming the sectional curvature to be nonnegative, [7] establishes a Harnack inequality for solutions of nondivergence form elliptic equations with measurable coefficients which extends the classical Euclidean theory of Krylov-Safonov. It applies to operators in Riemmanian manifolds of the form

$$Lu = \operatorname{tr}(A(x)D^2u),$$

where D^2u is the Riemannian Hessian and A is a section of the uniformly positive definite symmetric endomorphisms of the tangent bundle of M.

The Harnack inequality obtained in [7] is rescaled invariant. Thus, it has as a corollary the following *Liouville theorem*:

Theorem 5.14. *Let M be a Riemannian manifold with nonnegative sectional curvature. Let u be bounded from below, and be a smooth solution of $Lu = 0$ in all of M. Then, u is constant.*

In an interesting work, Seick Kim [39] has extended the nondivergent techniques of [7]. His more general results give in particular a new (nondivergent) proof of the classical theorem of S.T.Yau on the Laplace-Beltrami operator: "bounded harmonic functions on a whole manifold with nonnegative *Ricci* curvature are constant".

After [7], McCann [46] used the generalized Legendre transform (5.22) and its associated map (5.23) to prove existence of optimal transport maps on Riemannian manifolds —see subsection 2.7 above for the notion of optimal transport map in Euclidean space.

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