Stable solutions to semilinear elliptic equations are smooth up to dimension 9

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Joint work with Alessio Figalli, Xavier Ros-Oton, and Joaquim Serra. Acta Math. 2020

- Semilinear elliptic PDEs: $-\Delta u = f(u)$ in $\Omega \subset \mathbb{R}^n$, bdd domain

1st variation

Energy: $E_\Omega(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u)$, $F' = f$

2nd variation is $-\Delta - f'(u)$ = linearized operator at u for the equation $-\Delta u = f(u)$

it is nonnegative iff $-\Delta - f'(u) \geq 0$

iff $\int_\Omega f'(u)\xi^2 \leq \int_\Omega |\nabla \xi|^2 \quad \forall \xi \in C_c^\infty(\Omega)$ ← Def. of stability

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→ Competitors $u + \varepsilon\xi$ have all same boundary values as u

→ Our interest: nonlinearities f superlinear at $+\infty$ & $f \geq 0$

$\Downarrow$

NO absolute minimizer exists

$$E_\Omega(t\varphi) = t^2 \int_\Omega \frac{1}{2}|\nabla\varphi|^2 - \int_\Omega F(t\varphi) \xrightarrow[t\to+\infty]{} -\infty \quad \left(F(t\varphi) \gg t^2\varphi^2\right)$$

- The Barenblatt-Gelfand problem 1963 :

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f(0) > 0$, nondecreasing, convex, & superlinear at $+\infty$.

Model nonlinearities: $f(u) = e^u$ (combustion theory)

$f(u) = (1+u)^p$, $p > 1$

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with $f(0) > 0$, nondecreasing, convex, & superlinear at $+\infty$.

- Then, $\exists \lambda^* \in (0, +\infty)$ & $0 < \lambda < \lambda^* \Rightarrow \exists u_\lambda > 0$ stable classical (L^∞) sol'n
  - $u_\lambda \nearrow u^*$ as $\lambda \nearrow \lambda^*$

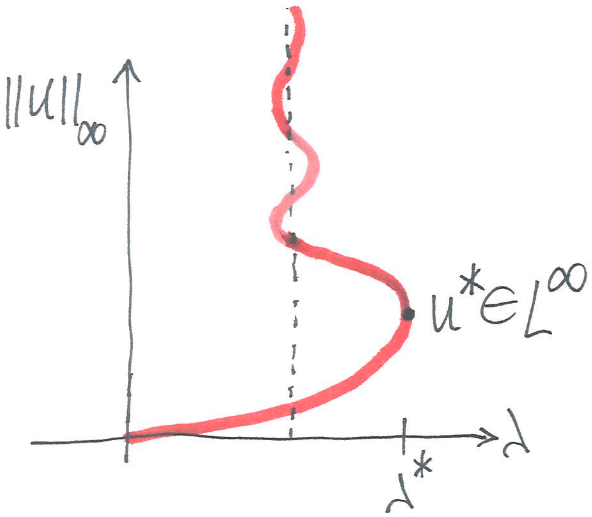
    $\hookrightarrow$ $u^* \in L^1(\Omega)$ is a distributional stable solution for $\lambda = \lambda^*$

    u^* = the extremal solution of the pb.
- $\nexists$ solutions for $\lambda > \lambda^*$

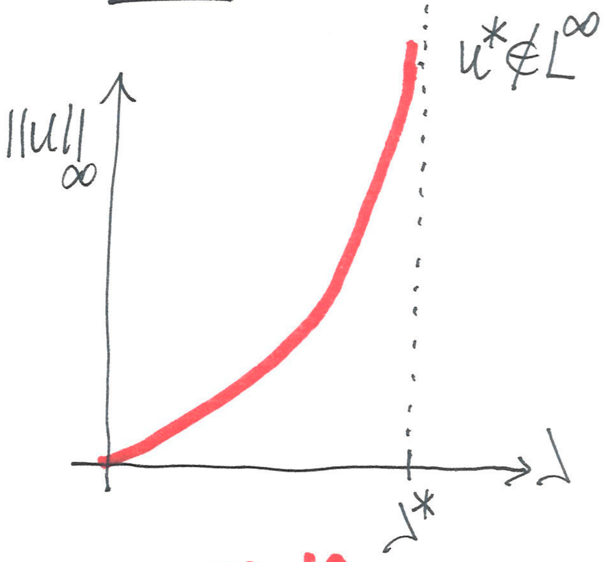
Model nonlinearities: $f(u) = e^u$ (combustion theory)

$f(u) = (1+u)^p$, $p > 1$

• [Joseph-Lundgren '72] $f(u) = e^u$ & $\Omega = B_1$ (RADIAL case) :



$3 \leq n \leq 9$



$n \geq 10$

▪ ODE techniques

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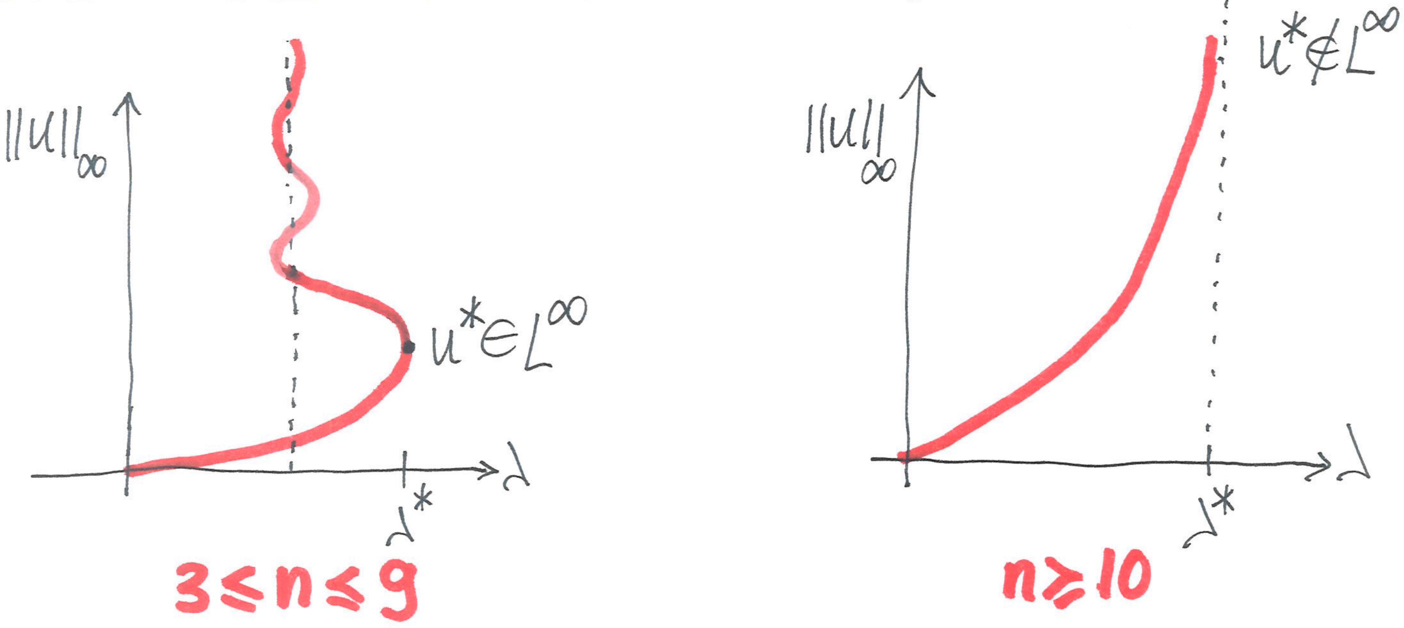

▪ ODE techniques

▪ Explicit singular solution:

$u(x) = -2 \log|x| \in W_0^{1,2}(B_1)$

solves $-\Delta u = 2(n-2)\, e^u$ in B_1, $n \ge 3$

Linearized operator $= -\Delta - 2(n-2)\frac{1}{|x|^2}$

(Hardy's ineq) $\to$ u stable $\Leftrightarrow 2(n-2) \le \frac{(n-2)^2}{4} \Leftrightarrow n \ge 10$

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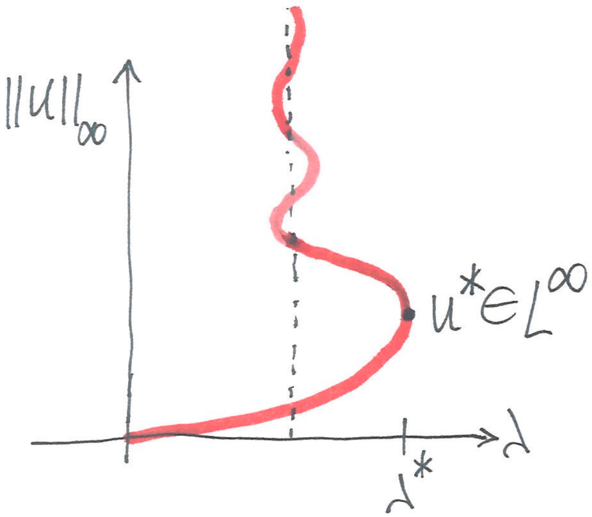

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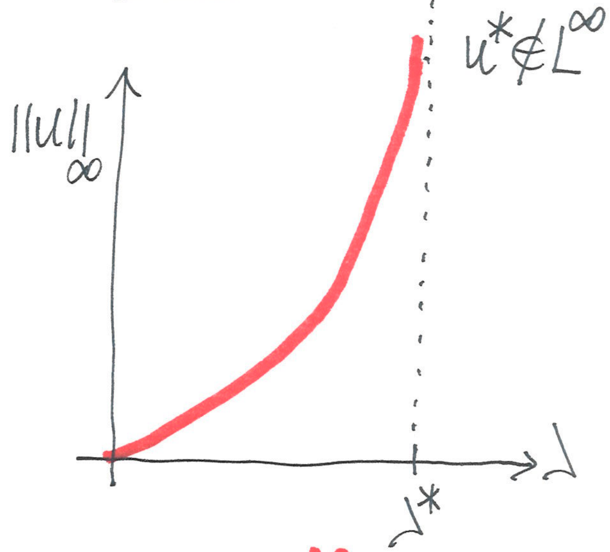

$n \geq 10$

- ODE techniques
- Similar for $f(u)=(1+u)^p$

$\downarrow$

explicit solutions

$u(x) = |x|^{-\alpha_p} - 1$

$(\alpha_p > 0)$

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(Hardy's ineq) $\rightarrow$ u stable $\Leftrightarrow$ $2(n-2) \leq \dfrac{(n-2)^2}{4}$ $\Leftrightarrow$ $n \geq 10$

- **Questions:** When is $u^* \in L^\infty(\Omega)$?
  When are $W_0^{1,2}$ stable solutions bounded ?

- For general solutions, L^∞ estimates exist for f subcritical or critical : $|f(u)| \leq C(1+|u|)^p$, $p \leq \frac{n+2}{n-2}$

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- PDE analogue of "regularity of stable minimal surfaces in $\mathbb{R}^n$":
  - → Not true for $n \geq 8$
  - → True for $n=3$ ([Fischer-Colbrie & Schoen '80] [DoCarmo & Peng '79])
  - → Open pb for $4 \leq n \leq 7$!!

(→ Known for $n \leq 7$ for minimizing minimal surfaces)

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- 1st result $\forall \Omega$: [Crandall-Rabinowitz '75]
  $u^* \in L^\infty(\Omega)$ if $n \leq 9$ and $f(u) \sim e^u$ or $f(u) \sim (1+u)^p$

- [Brezis-Vázquez '97] Is it always $u^* \in W_0^{1,2}(\Omega)$?
- [Brezis '03] Is there something "sacred" about dim 10 ?
  Is it possible to construct a singular stable sol'n for $n \leq 9$, in some domain & for some f ?

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- [Villegas '13] $u^* \in L^\infty(\Omega)$ if $n \leq 4$; $u^* \in W_0^{1,2}(\Omega)$ if $n \leq 6$

- [Cabré & Ros-Oton '13] L^∞ if $n \leq 7$ & Ω of double revolution
- [Cabré-Sanchón-Spruck '16] L^∞ if $n \leq 5$ & $f'/f^{1+\varepsilon} \leq C(\varepsilon)$ $\forall \varepsilon > 0$

- [Cabré, Figalli, Ros-Oton, Serra '19]

Thm 1 $u \in C^2(B_1)$ stable sol'n of $-\Delta u = f(u)$ in B_1 & $f \geq 0 \implies$

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2})} \leq C(n)\, \|u\|_{L^1(B_1)} \qquad (\gamma = \gamma(n) > 0)$$

& if $n \leq 9$ then $\|u\|_{C^\alpha(\overline{B}_{1/2})} \leq C(n)\, \|u\|_{L^1(B_1)}$ $(\alpha = \alpha(n) > 0)$.

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Corol 1 $L^\infty(\Omega)$ estimate for $n \leq 9$ (if $f \geq 0$) and any stable sol'n of $\begin{cases} -\Delta u = f(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$ if Ω is bdd convex C^1 domain.

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Thm 2 Ω bdd C^3 domain, $f \geq 0$, $f' \geq 0$, $f'' \geq 0$. $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ stable sol'n of $\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$ $\Rightarrow$

$$\|\nabla u\|_{L^{2+\gamma}(\Omega)} \leq C(\Omega) \|u\|_{L^1(\Omega)} \qquad (\gamma = \gamma(n) > 0)$$

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Corol 2 Ω bdd C^3 domain $\Longrightarrow$ $\begin{cases} u^* \in W_0^{1,2+\gamma}(\Omega) & (\gamma = \gamma(n) > 0) \\ \text{if } n \leq 9,\ u^* \in L^\infty(\Omega). \end{cases}$

Thm 3 Sharp Morrey $M^{p,\beta}(\Omega)$ estimates for stable sol'ns when $n \geq 10$.

RELATED WORK:

- **p-Laplacian** $\quad -\Delta_p u = f(u) \quad , \quad 1<p<2$
  - [Cabré-Miraglio-Sanchón '20] Optimal result for $p>2$: regularity if $n < p + \frac{4p}{p-1}$.
  - Optimal result is open for $1<p<2$.
- **Fractional Laplacian** $\quad (-\Delta)^s u = f(u) \quad , \quad 0<s<1$
  - Optimal dimensions: open even in the radial case

    $\downarrow$

    involved relation on Γ-function: only known for $f(u)=e^u$ in convex symmetric domains [Ros-Oton '14]

- PROOFS

$$\Delta u + f(u) = 0 \qquad \text{(EQUATION)}$$

$$\downarrow$$

$$\Delta + f'(u) \qquad \text{(LINEARIZED OPERATOR} \leq 0\text{)}$$

$$\downarrow$$

$$\int_\Omega f'(u)\,\xi^2 \leq \int_\Omega |\nabla \xi|^2 \quad \forall \xi \in C^1_c(\Omega) \qquad \text{(STABILITY)}$$

$$\Big\downarrow \; \xi = c \cdot \eta \qquad \text{with } \eta_{|\partial\Omega} \equiv 0.$$

$$\int_\Omega c\,\underline{(\Delta c + f'(u)c)}\,\eta^2 \leq \int_\Omega c^2 |\nabla \eta|^2 .$$

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[Cabré-Capella '05]
motivated by
Simons' lemma on
minimal cones
↑
c = ‖second fund. form‖

- **Proofs** $\xi = c \cdot \eta \quad \Longrightarrow \quad \int_\Omega c(\Delta c + f'(u)c)\,\eta^2 \le \int_\Omega c^2 |\nabla \eta|^2 . \quad (\eta_{|\partial\Omega} \equiv 0)$

- [Crandall-Rabinowitz] & [Nedev] : $\underline{\xi = h(u)}$

- [Cabré-Capella] $(\Omega = B_1)$: $\xi = \underbrace{r u_r}_{= c = x \cdot \nabla u} \cdot \underbrace{r^{-a} \varphi}_{= \eta = |x|^{-a} \cdot \varphi}$, φ cut-off near ∂B_1

- [Cabré '10] $(n \le 4)$: $\xi = \underbrace{|\nabla u|}_{= c} \cdot \underbrace{g(u)}_{= \eta}$

• **Proofs** $\xi = c\cdot\eta \qquad \Rightarrow \int_\Omega c(\Delta c + f'(u)c)\,\eta^2 \leq \int_\Omega c^2|\nabla\eta|^2. \qquad (\eta|_{\partial\Omega} \equiv 0)$

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& $c = |\nabla u|$

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For our interior result ($n\le 9$) we will use both
$c = x\cdot\nabla u$
& $c = |\nabla u|$

$$\underline{(\Delta + f'(u))(x\cdot\nabla u)} = 2\Delta u$$

&

$$\underline{(\Delta + f'(u))|\nabla u|} = \frac{1}{|\nabla u|}\left\{ \sum_{i,j} u_{ij}^2 - \sum_i \Big(\sum_j u_{ij}\frac{u_j}{|\nabla u|}\Big)^2 \right\}$$

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Curvature of level sets

+

Michael-Simon Sobolev ineq.

Using $\xi = c\eta = (x\cdot\nabla u)\,\eta(x) \rightsquigarrow \int_\Omega (x\cdot\nabla u)\, 2\Delta u\, \eta^2$ $\rightsquigarrow$ Pohozaev trick

Lemma 1 $\forall n\ \forall f\ \forall u$ stable sol'n $\forall \eta \in C^1_c(B_1) \implies$

$$\int_{B_1} \underbrace{\{(n-2)\eta + 2x\cdot\nabla\eta\}}_{(0)} \eta\, \underline{|\nabla u|^2} - 2\,\underline{(x\cdot\nabla u)}\ \underline{\nabla u\cdot\nabla(\eta^2)}_{(1)}$$

$$- \underbrace{|x\cdot\nabla u|^2}_{(2)} |\nabla\eta|^2 \le 0.$$

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$$\xi = (x\cdot\nabla u)\,\underbrace{|x|^{\frac{2-n}{2}}\varphi(x)}_{=\eta(x)}$$

so that $\{\cdots\} \ge 0$ ⓪

Using $\xi = c\eta = (x\cdot\nabla u)\,\eta(x)$ $\rightsquigarrow$ $\int_\Omega (x\cdot\nabla u)\, 2\Delta u\, \eta^2$ $\rightsquigarrow$ Pohozaev trick

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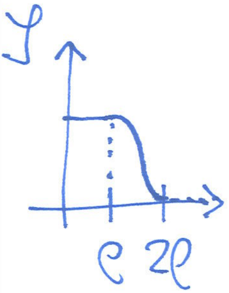

$\textcircled{1}$ & $\textcircled{2}$ $\longrightarrow$ $2(n-2) - \dfrac{(n-2)^2}{4} = \dfrac{1}{4}\{8(n-2)-(n-2)^2\}$

$$= \frac{1}{4}(n-2)(10-n)$$

$$\underline{\frac{1}{4}(n-2)(10-n)}\int_{B_\rho} |x|^{2-n} u_r^2 \le C\int_{B_{2\rho}\setminus B_\rho} |x|^{2-n}|\nabla u|^2$$

NOTE: $\displaystyle\int_{B_{1/2}} |x-y|^{2-n} \left| \nabla u(x) \cdot \frac{x-y}{|x-y|} \right|^2 dx \leq C \qquad \forall y \in B_{1/2}$

$\hookrightarrow$ $u \in BMO$ if $n \leq 9$

easy

WE HAVE:

$$\int_{B_\rho} |x|^{2-n} u_r^2 \leq C \int_{B_{2\rho} \setminus B_\rho} |x|^{2-n} |\nabla u|^2$$

NOTE: $\displaystyle\int_{B_{1/2}} |x-y|^{2-n} \left|\nabla u(x)\cdot\frac{x-y}{|x-y|}\right|^2 dx \leq C \qquad \forall y\in B_{1/2}$

$\xrightarrow[\text{easy}]{}$ **$u\in BMO$** **if** $n\leq 9$

WE HAVE:

$$\boxed{\int_{B_\rho} |x|^{2-n} u_r^2 \leq C \int_{B_{2\rho}\setminus B_\rho} |x|^{2-n} |\nabla u|^2}$$

If we had $\displaystyle\int_{B_{2\rho}\setminus B_\rho} |x|^{2-n}|\nabla u|^2 \leq C' \int_{B_{2\rho}\setminus B_\rho} |x|^{2-n} u_r^2$, then $\rightarrow$

$$\rightarrow \int_{B_\rho} |x|^{2-n} u_r^2 \leq C'' \int_{B_{2\rho}\setminus B_\rho} |x|^{2-n} u_r^2$$

adimensional quantity

$$\rightarrow \int_{B_\rho} |x|^{2-n} u_r^2 \leq \frac{C''}{1+C''} \overbrace{\int_{B_{2\rho}} |x|^{2-n} u_r^2}$$

$\Longrightarrow$ Algebraic decay & **Hölder continuity of u**

We would like

$$\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 \leq C(n) \int_{B_{1/2}\setminus B_{1/4}} u_r^2 . \qquad (*)$$

May it be true?

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May it be true ?

If false, in the extreme case we would have

$$\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 = 1 \quad \& \quad \int_{B_{1/2}\setminus B_{1/4}} u_r^2 = 0$$

$\rightarrow$ u is 0-homogeneous

$\Downarrow$ $-\Delta u = f(u) \geq 0$

u is a superharmonic f'cn on the sphere S^{n-1}

$u = ctt \Leftarrow$

CONTRADICTION

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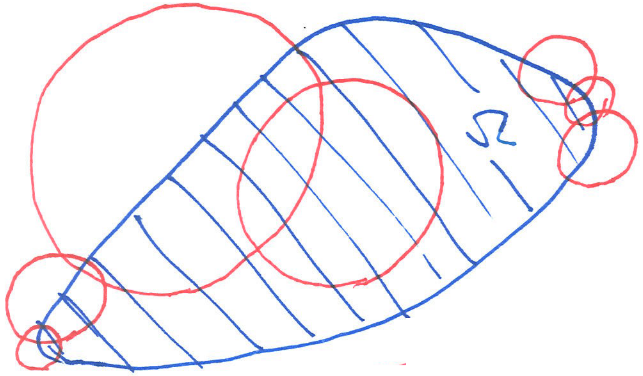
$u = ct \Leftarrow$

CONTRADICTION

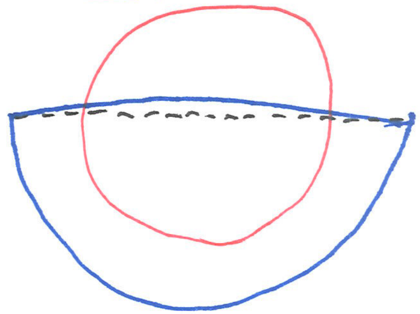
$\to$ We prove $(*)$ (under a doubling assumption that suffices) by COMPACTNESS using the higher integrability estimate

$c = |\nabla u| \Longrightarrow \|\nabla u\|_{L^{2+\gamma}} \leq C(n) \|\nabla u\|_{L^2}$

- **Boundary regularity**



Almost half-balls



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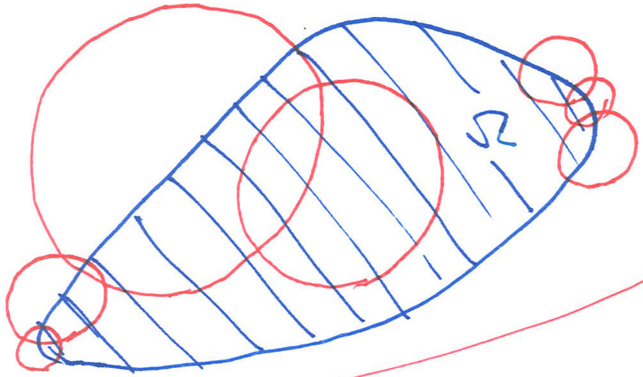

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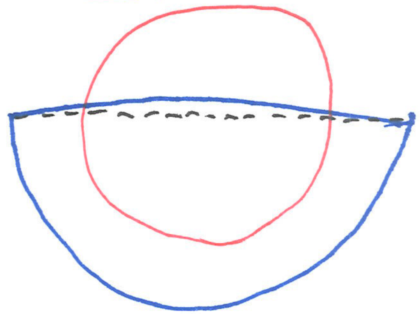
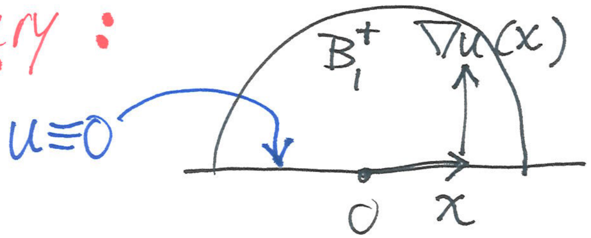


Simplest case:

Half-balls ; flat boundary :

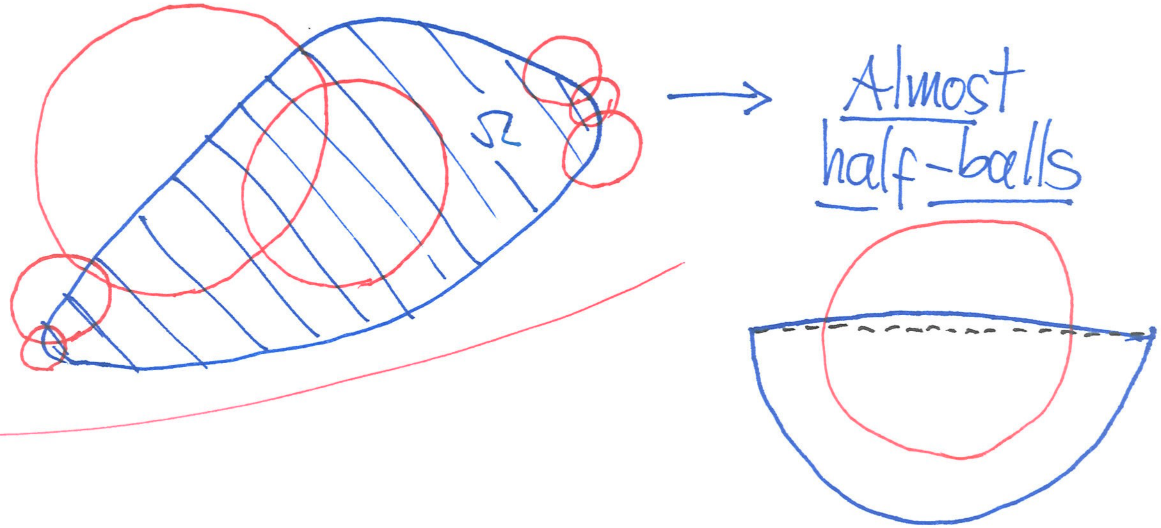


$\xi(x) = (x \cdot \nabla u)|x|^{\frac{2-n}{2}} \varphi(x)$ vanishes on the flat bdry ☺

$$\Downarrow$$

$$(n-2)(10-n)\int_{B_\rho^+} |x|^{2-n} u_r^2 \leq C \int_{B_{2\rho}^+ \setminus B_\rho^+} |x|^{2-n} |\nabla u|^2$$

• Boundary regularity



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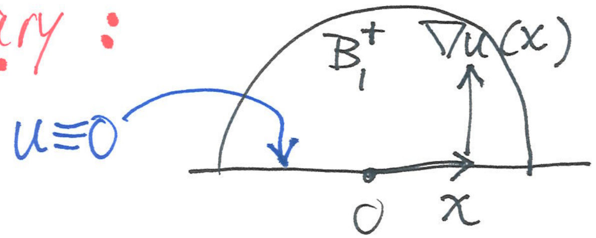

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We ask ¿ $\exists u$, $u=0$ on $\{x_n=0\}$, $\Delta u<0$ in $\{x_n>0\}$,

$$\int_{B_2^+\setminus B_1^+} |\nabla u|^2 = 1 \quad \& \quad \int_{B_2^+\setminus B_1^+} u_r^2 = 0 \; ?$$

Yes ☹ $u(r,\theta)=\sin\theta$

key remark: u cannot solve $-\Delta u = f(u)$ if $u = u(\theta)$

0 homogeneous

-2 homogeneous

Question: Can one pass to the limit the condition $-\Delta u = f(u)$ **?**

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Thm 4 Let u_k be stable sol'ns of $-\Delta u_k = f_k(u_k)$ in $U \subset \mathbb{R}^n$ open,

with $f_k' \geq 0$, $f_k'' \geq 0$; $u_k \in W^{1,2}_{loc}(U)$

Then $u_k \to u$ in $L^1_{loc}(U)$.

$u \in W^{1,2}_{loc}(U)$ is a stable solution of $-\Delta u = f(u)$ in U

for some f nondecreasing and convex, $f : (-\infty, M) \to \mathbb{R}$.

Thanks for your attention